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On the Logical Foundations of the Jauch–Piron Approach to Quantum Physics

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We make a critical analysis of the basic concepts of the Jauch-Piron (JP) approach to quantum physics. Then, we exhibit a formalized presentation of the mathematical structure of the JP theory by introducing it as a completely formalized syntactic system, i.e., we construct a formalized language $\mathbf{L}_e$ and formally state the logical-deductive structure of the JP theory by means of $\mathbf{L}_e$. Finally, we show that the JP syntactic system can be endowed with an intended interpretation, which yields a physical model of the system. A mathematical model endowed with a physical interpretation is given which establishes (in the usual sense of the model theory) the coherence of the JP syntactic system.

1. INTRODUCTION

The Jauch-Piron (JP) approach to quantum physics (QP) (here also called JP theory) is widely known. Its roots lie in the early Birkhoff and Von Neumann (1936) paper on the logic of quantum mechanics, and it has been extensively discussed (Piron, 1964, 1972, 1976a,b, 1977, 1978, 1981; Jauch, 1968, 1971; Jauch and Piron, 1969; 1970). Recently, it has been developed along original lines and amalgamated with the "operational approach" to QP proposed by Foulis *et al.* (1983).

The JP theory has been criticized from several points of view (e.g., Mielnik, 1976; Cooke and Hilgevoord, 1981; Hadjisavvas *et al.*, 1980, 1981; Thieffine, 1983; Hughes, 1981). In particular, Mielnik (1976) and, more recently, Hughes (1981) have censured the lattice operations introduced by Piron, while Hadjisavvas *et al.* (1980, 1981; Thieffine, 1983) have argued about the coherence (which they prove) and the possibility of giving a physical interpretation of the axioms (which they deny) of the theory.

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An extensive answer to the latter criticism has been given by Foulis and Randall (1984), making reference to the background established in Foulis *et al.* (1983) (suitably condensed and modified); yet, their answer was rejected by Hadjisavvas and Thieffine (1984) and the polemics do not seem exhausted.

Our opinion about the matter can be synthetized as follows.

1. Most criticism to the JP theory can be proved erroneous or based on a miscomprehension, especially those raised against the late formulations of the theory, even if Foulis *et al.* (1983) is not taken into account. Yet, the usual presentations of the JP approach to QP are unsatisfactory from an epistemological viewpoint, and contain semantical ambiguities which have fostered misunderstandings. Thus, a critical discussion of the JP theory and a more formal presentation of it seem desirable.

2. The development of the JP theory presented in Foulis *et al.* (1983) is not strictly needed in order to defend the theory; moreover, it seems to be exposed to many of the epistemological objections which can be raised against the standard formulation of the JP theory [in particular, the existence of "hidden" axioms in Foulis and Randall (1984), hence a coherence problem, is not recognized]. A discussion about this subject has been already carried out by Cattaneo and Nisticò (1986).

Thus, rather than entering the aforesaid polemics and discussing the above criticisms in detail, we intend to make a critical analysis of the foundations of the JP approach, mainly referring to the Piron presentation of it and dispensing with Foulis *et al.* (1984). This analysis endows us with the necessary background in order to propose a formalized presentation of the theory which takes into account some relevant epistemological distinctions, so as to eliminate many ambiguities. In this framework, we intend to give a new proof of the (relative) coherence of the JP approach by means of a mathematical model which has a direct physical interpretation in QP [so that it gives a possible answer to a challenge in Hadjisavvas *et al.* (1980).

More specifically, we proceed as follows.

First, we schematize in Section 2 the essentials of the JP approach to QP according to the Piron presentation.

Second, we discuss in Section 3 some epistemological collapse in this presentation, which obscures, in particular, the distinction between the mathematical structure of the theory and its interpretation; the role of some "physical definitions" introduced by Piron, and the existence of some hidden axioms besides the axioms which are explicitly stated; the convenience of introducing some new predicates, like "false" and "indeterminate," which clarify, by opposition, the meaning of Piron's predicate "true"; the structures of the sets of questions and of propositions, and their differences; the improper transfer of results from one of these structures to the other, which has given origin to some apparent "paradoxes" in the JP theory.

Third, we comment in Section 4 on some advantages of a formalized presentation of the basic structures of the JP approach. Then, we construct a syntactic scheme for a language $\mathbf{L}_e$ which formalizes Piron's mathematical language, and formulate all the basic axioms, some preliminary results, and the fundamental theorem of the JP theory by means of $\mathbf{L}_e$. We also add some technical comments on our language $\mathbf{L}_e$, show that it can be suitably enlarged so as to embody the new predicates introduced in Section 3, and note the impossibility of weakening some axioms of the theory without losing some of its essential features.

Fourth, we propose in Section 5 an intended interpretation of $\mathbf{L}_e$, that is, an interpretation on the physical domain considered by JP; from this, an interpretation of axioms, definitions, and results is derived (it should be observed that neither $\mathbf{L}_e$ nor the "hidden" axioms of the JP theory nor Piron's axioms C, P, A express the difference between classical and quantum physics; this follows from well-known features of the JP approach and holds unchanged in the formalized theory).

Fifth, we discuss in Section 6 the coherence problem for the set of axioms introduced in Section 4 and prove the (relative) coherence of the JP theory by giving a mathematical model for its mathematical structure; we also discuss the physical meaning of our model (it should not be confused with the usual Hilbert model for propositional systems given by Piron, which would not be sufficient for our purposes), and show that some properties of the question structure which have been evidenced by other authors (Hadjisavvas *et al.*, 1980, 1981; Thieffine, 1983) are realized in our model.

Finally, we give in the Appendix a formal proof of the fundamental theorem of the JP theory, thus showing that the set of "hidden" axioms explicated in Sections 3 and 4, together with Piron's set of axioms, is adequate in order to obtain all the basic results of the theory.

2. THE JP APPROACH

In the JP approach to questions and propositions, as presented mainly in Piron's papers, some metatheoretical statements are premised which express Piron's assent to the "program of realism," formulated as follows:

> (the aim of physics is) to give a complete description of each individual system as it is in all its complexity,

and which coherently define what is meant with the expression "physical system" (Piron, 1976a):

> by a *physical system* we mean a part of real world thought of as existing in space-time and external to the physicist.

Having thus established an epistemological standpoint, the *basic concept* of *question* is introduced by observing that the statements of the physicist about a physical system can be checked by experiments, and that such a test consists, in general, of a measurement, the result of which is expressed by "yes" or "no."

PD2.1. We shall call a question every experiment leading to an alternative of which the terms are "yes" or "no" (Piron, 1976a).

Thus, schematically, a question consists of (a) a measuring apparatus, (b) instruction for its use, and (c) a rule interpreting the possible results in terms of "yes" or "no" (Piron, 1977).

Then, the following *physical definitions* are introduced.

PD2.2. There exists a trivial question, which we shall denote as I, and which consists in nothing other than measuring anything (or doing nothing)and stating that the answer is "yes" each time (Piron, 1976a).

PD2.3. If α is a question, we denote by α^{ν} the question obtained by exchanging the terms of the alternative (Piron, 1976a).

It follows from PD2.3 that, if the result of α for a single individual system is "yes," then the result of α^{ν} is "no," and vice versa; thus, α^{ν} can be measured by the same equipment used for the measurement of α.

PD2.4. If $\{\alpha_i\}$ is a family of questions, we denote by $\prod_i \alpha_i$ the question defined in the following manner: one measures an arbitrary one of the α_i and attributes to $\prod_i \alpha_i$ the answer thus obtained (Piron, 1976a).

It follows from PD2.4 that (a) the measuring apparatus for $\prod_i \alpha_i$ is the set of the measuring apparatuses for the α_i; (b) the instructions for the use of the apparatus are: (1) one chooses at random one of the α_i in the family and (2) one performs the corresponding experiment on a single individual physical system obtaining one of the two possible results, either "yes" or "no"; (c) the rule for interpreting the obtained result is that of attributing it to $\prod_i \alpha_i$. It must be stressed that the above physical definitions involve measurements of questions on a single individual physical system.

Now, Piron introduces the following definition of the diadic *predicate* "true":

PR2.1. When the physical system has been prepared in such a way that the physicist may affirm that in the event of an experiment the result will be "yes," we shall say that the question is certain or that the question is "true" (Piron, 1976a).

Then, the following *derived definitions* are introduced.

D2.1. For certain pairs of questions β, γ, one may have the following relation:

If the physical system is prepared in such a way that β is true, then one is sure that γ is true.

We denote it as $\alpha < \beta$ and read it as "β less than γ" (Piron, 1972). (Hence, $<$ is a quasiorder relation.)

D2.2. For every pair of questions β, γ the relation $<$ defines an equivalence (Piron, 1972)

$$\beta \sim \gamma \quad \text{iff} \quad \beta < \gamma \quad \text{and} \quad \gamma < \beta$$

D2.3. We define a proposition as an equivalence class of questions, and denote by b the equivalence class containing the question β; i.e., $b = \{\gamma | \gamma \sim \beta, \gamma \text{ a question}\}$ (Piron, 1972).

D2.4. If [the question] β is true, then any $\gamma \sim \beta$ is true. Hence, we can say that the proposition [$b = \{\gamma | \gamma \sim \beta, \gamma \text{ a question}\}$] is *true* iff any and therefore all of its questions are true (Piron, 1972). So, there is a one-to-one correspondence between *propositions* and [physical] *properties* [$\cdots$]. If one given system has been prepared in such a way that we can affirm that in the event of the experiment the expected result would be certain, we will say that the corresponding property is an *actual property* of the system, in opposition to the other properties, which are only potential (Piron, 1981).

D2.5. We denote by $\mathcal{L}$ the set of propositions defined for a given physical system (Piron, 1972).

D2.6. A (partial) ordering relation is defined in $\mathcal{L}$ (Piron, 1972)

$$b < d \quad \text{iff} \quad \beta < \delta \quad \text{with} \quad \beta \in b \quad \text{and} \quad \delta \in d$$

By making use of the basic concepts and definitions above, Piron proves the fundamental theorem of the theory.

Theorem 2.1. The set of propositions $\mathcal{L}$ is a complete lattice (Piron, 1976a).

Then, Piron introduces some further derived symbols, the definitions of which we make here explicit as follows.

D2.7. We denote by $\cup$ and $\cap$ the meet and join in $\mathcal{L}$, respectively.

D2.8. We denote by O the question I^{ν}.

D2.9. We denote by $\mathbb{1}$ and $\mathbb{0}$ the propositions that contain I and O, respectively.

By making use of the above symbols, the following definition of compatible complement is stated in the Piron presentation of the theory.

D2.10. Let b be a proposition and c a complementary proposition for b:

$$b \cup c = \mathbb{1} \qquad \text{and} \qquad b \cap c = \mathbb{0}$$

We shall say that c is a *compatible complement* for b if there further exists a question β such that

$$\beta \in b \qquad \text{and} \qquad \beta^{\nu} \in c$$

that is, if there exists in the equivalence class b a question β such that β^{ν} is in the equivalence class c (Piron, 1976a).

Now, Piron introduces two fundamental axioms of the theory.

Axiom C. For each proposition there exists at least one compatible complement (Piron, 1976a)

Axiom P. If $b < c$ are propositions of $\mathscr{L}$ and if b' is a compatible complement for b and c' a compatible complement for c, then the sublattice generated by $\{b, b', c, c'\}$ is a classical propositional system, that is to say, a distributive lattice (Piron, 1976a).

From axioms C and P one easily deduces some further basic results of the theory. First, for every $a \in \mathscr{L}$, there exists in $\mathscr{L}$ one compatible complement only, which will be denoted by a'. Second, the mapping

$$': a \in \mathscr{L} \mapsto a' \in \mathscr{L}$$

is an orthocomplementation in $\mathscr{L}$. Third, the lattice $\mathscr{L}$ is weakly modular.

Finally, Piron states the third fundamental axiom of the theory as follows.

Axiom A:
A_1. If b is a proposition different from $\mathbb{0}$, there exists an atom $p < b$.
A_2. If p is an atom and if $p \cap b = \mathbb{0}$, then $p \cup b$ covers b (Piron, 1976a).

A complete lattice satisfying C, P, and A is called by Piron a propositional system.

Thus, the basic definitions, axioms, and results that are needed in order to build up the theory are stated. Our exposition of the JP theory will therefore end here; indeed, we are interested in these basic elements only in the present paper.

3. SOME CRITICAL REMARKS ABOUT THE JP APPROACH

As we have anticipated in the Introduction, we would like to make some comments and a critical analysis in the present section on some of the definitions and results in Section 2, and on the JP approach as a whole.

Our discussion will also give an answer to some arguments that have been raised against the JP approach to QP.

3.1. We begin with some epistemological remarks on the JP theory. Our arguments will be better understood if we recall that, according to a largely accepted epistemological conception, usually called the *standard epistemological conception*, or *received viewpoint*, e.g. Braithwaite (1953), Carnap (1956, 1966), Nagel (1961), and Hempel (1965), every physical theory consists of two separate parts, as follows: (a) a *theoretical structure* (or *apparatus*) and (b) a set of *correspondence* or *epistemic* rules.

Furthermore, the theoretical structure can be split into the following components: (a_1) a *mathematical structure* or *calculus*, which is a purely syntactical axiomatic deductive system, and (a_2) an *intended physical interpretation.*[4]

Finally, the mathematical structure of the theory may be interpreted over a domain of mathematical objects, thus providing a mathematical model of the theory.

The above distinctions, which are epistemologically relevant, are often ignored in practice, both when stating a theory and when discussing it. This has been a primary source of confusion and pseudoparadoxes. Thus, in most physical theories the correspondence rules are not explicitly stated or mentioned, since they are implicitly (and incorrectly!) identified with some physical intended interpretation (it should be noted that, according to an instrumentalist viewpoint, the only elements which are logically needed when setting up a physical theory are the mathematical structure and the correspondence rules, while any intended physical interpretation has a purely euristic role). Furthermore, no identification is legitimate between

[4]The intended physical interpretation and the correspondence rules provide two epistemologically different interpretations of the mathematical structure. More specifically, the former is a *complete* and *direct* interpretation of the calculus over a domain of physical objects, usually having the status of theoretical entities, which starts from the axioms and yields an isomorphic physical model of the mathematical structure; the latter provides a *partial* and *indirect* empirical interpretation of the calculus over a domain of observative entities, which starts from the (nontrivial) theorems. The differences between these two kinds of interpretation, which generally correspond to the canonical differences between a theory and its models, have been analyzed in detail by Braithwaite (1953, 1960); for further discussion on the subject, see also Duhem (1914) and Groenewold (1961). We remark that the word "intended" here outlines the privileged role of the interpretation considered in (a_2) with respect to the other physical models of the calculus which are abstractly possible.

the mathematical structure and its physical model; yet, these often are not distinguished, especially by those authors who share a realistic attitude. Besides, the mathematical structure must also be distinguished from its mathematical models (e.g., the mathematical structure of QP must not be confused with the Hilbert space model for QP); it may be interesting to observe that in many cases a mathematical model may be obtained by considering some current privileged mathematical representation of the physical objects considered in the intended physical interpretation.

Let us come now to the JP theory. First, JP never explicitly deal with (b). We deem that the JP interpretation is an intended interpretation in the sense specified above (it is indeed direct and complete; furthermore, it ranges over infinite sets of physical objects, so that these necessarily have the status of theoretical entities). Second, JP do not distinguish between (a_1) and (a_2): the basic mathematical symbols and axioms of the theory are introduced together with their physical interpretation (of course, this is done by making use of a metalanguage, which consists of a part of the natural language enriched with suitable technical symbols).

In addition to the above general remarks, we observe that JP never completely axiomatize their theory (it will be shown that Piron's presentation contains four hidden axioms) so that its basic assumption are not quite unambiguous. Finally, we also note that Piron's assumption about the existence of a one-to-one correspondence between propositions and properties (see D2.4) does not fit exactly with the usual meaning, in logic as well as in physics, of the words "proposition" and "property"; it is apparent, in particular, that more than one property can generally be associated to a given proposition a, in the sense of Piron (more precisely, all those properties which are physically equivalent to some property characterizing a).[5]

Thus, the JP formulation of their theory, while undoubtedly adequate for physicist's use, is unsatisfactory from an epistemological viewpoint; significantly, as we have already observed, a number of miscomprehensions have arisen about its basic concepts.

3.2. We now focus on the physical definitions introduced in Section 2 and on the specific axioms of the JP theory.

[5]We recall that some primitive concepts in the JP approach are expressed in terms of more fundamental concepts in the formalism presented in Foulis *et al.* (1983); here, in particular, a mathematical representation is given for a physical system, the concept of state is introduced (this being in agreement with our remark at the beginning of Section 3.3), etc. In this framework, a sharp distinction is made [see also Foulis and Randall (1984)] between operationally testable propositions, which correspond to questions, and properties, which still correspond to JP propositions in the sense specified by D2.4, and do not necessarily admit an operational test. We will not comment about these (rather problematical) correspondences here; we will limit ourselves to the observation that the above remark about the use of the word "property" seems to apply even to Foulis *et al.* (1983).

The definitions PD2.1–PD2.4 state a correspondence between symbols (such as α, $^\nu$, $\prod$) and objects of, or relations onto, a physical domain (the set of questions $\mathcal{Q}$ defined for a given physical system[6]); therefore, they may be classified as interpretation rules of the theory. From these definitions, together with PR2.1, the following results can be obtained.

A3.1. I is true.
A3.2. For any question α, $\alpha^{\nu\nu} = \alpha$.
A3.3. For any family of questions $\{\alpha_i\}$, $\prod_i \alpha_i$ is true iff each α_i is true; furthermore, $(\prod_i \alpha_i)^\nu$ is true iff $\prod_i \alpha_i^\nu$ is true.
A3.4. For any question α, α^ν true implies that α cannot be true.

The derivation of A3.1–A3.4 requires that the aforesaid symbols be identified with their representatives or, more correctly, that the domain of symbols be "isomorphic" (in some sense that will not be discussed here) to the physical domain. In a completely axiomatized (formalized or not) presentation of the theory, the above procedure should be reversed. Indeed, A3.1–A3.4 should be assumed as specific axioms, while the physical interpretation (which is an "intended" interpretation, as we have already commented on in Section 3.1) expressed by PD2.1–PD2.4 should be introduced afterward, in order to endow the theory with a physical meaning.

We observe that none of the statements A3.1–A3.4 (which, it must be stressed, express properties of the set of questions) is stated as an axiom in the JP approach. Rather, A3.1 appears as an obvious consequence of PD2.1, PD2.2, and PR2.1, while A3.2 and A3.4 trivially follow from PD2.1, PD2.3, and PR2.1 (A3.4, establishing a connection between the truth of the question α and α^ν). With regard to A3.3, which follows from PD2.1, PD2.4, and PR2.1, we note that the second part can be reformulated by writing that $(\prod_i \alpha_i)^\nu \sim \prod_i \alpha_i^\nu$. Now, Piron states that the stronger equation $(\prod_i \alpha_i)^\nu = \prod_i \alpha_i^\nu$ can be easily derived from PD2.1–PD2.4 (1976a). This derivation does not seem actually possible, as also suggested by the fact that the physical arrangements corresponding to $(\prod_i \alpha_i)^\nu$ and $\prod_i \alpha_i^\nu$, according to the rules stated by Piron, do not exactly coincide. Of course, the equality could be assumed (thus adding something new to the content of PD2.1–PD2.4); but such an assumption is not needed, as we will prove in the next section, and we will dispense with it from now on.

[6] The specification "defined for a given physical system" is ours, and it seems necessary in order to avoid a number of difficulties which arise (in particular, in the interpretation of $\prod$ and of the predicate true) if $\mathcal{Q}$ is an arbitrary set of questions, or the set of "all the conceivable questions." We think that this specification is coherent with the JP viewpoint, since reference to a given physical system is made in PR2.1 and D2.1, and the statement "defined for a given physical system" explicitly appears in D2.5, where the set $\mathcal{L}$ of propositions is introduced.

Thus, we have identified four statements which actually are axioms of the JP theory, though they were never recognized as such. Therefore, A3.1–A3.4 should be adjoined to Piron's axioms C, P, A (which mainly state properties of the set $\mathscr{L}$ of propositions) in order to obtain the set of all the basic axioms. Hence, a coherence problem occurs which cannot be solved by means of the standard models where axioms A3.1–A3.4 are not taken into account.

Moreover, the question arises whether no further axiom is hidden in the standard formulation of the JP approach. We show in the Appendix that the fundamental theorem of the JP theory, i.e., Theorem 2.1, can be deduced from A3.1–A3.4 only, by making use of classical inference rules. Since all the basic results of the JP theory follow from the structure of the propositional system of $\mathscr{L}$, which is established as a consequence of Theorem 2.1 and axioms C, P, A only, we conclude that our set of seven specific axioms is adequate in order to express all the main content of the JP theory.

3.3. Let us comment now on PR2.1. Indeed, this statement is basic in Piron's work, and the interpretation of it has been a remarkable source of quarrels.

We note first that the word "prepared" in PR2.1 implicitly introduces the set $\mathscr{S}$ of procedures which prepare single physical systems, each *preparation procedure* consisting of a macroscopic apparatus, which can also produce ensembles of identical physical systems under well-defined and repeatable conditions.

Second, we observe that a question can be tested on each single system by giving as a result of the measurement one of the two alternatives "yes" or "no"; but the word "certain" in PR2.1 implies that the question α can be said to be true with respect to a preparation x if and only if all the individuals of any ensemble prepared according to x give rise to the answer "yes" (we do not enter here into the epistemological problems raised by this kind of "certainty," which involve empirical induction and/or theoretical prediction).

Third, we notice that the notion of true given in PR2.1 suggests that a predicate "false" also be introduced by means of a similar definition.

PR3.1. When the physical system has been prepared in such a way that the physicist may affirm that in the event of an experiment the result will be "no," we shall say that the question is "false."

The definition of "false" can be equivalently formulated as follows (because of PD2.3).

> When the physical system has been prepared in such a way that the physicist may affirm that in the event of an experiment the result will be "yes" for the question α^{ν}, we shall say that the question α is "false."

By making use of this new predicate, the above result A3.3 can equivalently be restated as follows.

> A3.3′. For any family of questions $\{\alpha_i\}$, the product question $\prod_i \alpha_i$ is true iff each α_i is true and it is false iff each α_i is false.

Furthermore, A3.4 can be restated so as to obtain the following coherence condition for the predicates "true" and "false."

> A3.4′. There is no preparation with respect to which a question α is simultaneously true and false.

It also follows from PR2.1 and PR3.1 that for any question α some preparation could exist such that α is neither true nor false; then a third diadic predicate "indeterminate" can be introduced as follows.

> PR3.2. When the physical system has been prepared in such a way that in the event of an experiment neither the result "yes" nor the result "no" is certain, we shall say that the question is "indeterminate."

In the JP orthodox approach, the predicates "false" and "indeterminate" are not explicitly defined and, in our opinion, this has been a source of misunderstandings. Indeed, our above analysis of PR2.1 shows that we have two distinct levels of description: the first one, which pertains to each single physical system, leads, for every question, to the alternatives "yes" or "no"; the second one, which pertains to any preparation as a whole, leads, for every question, to the three alternatives "true," "false," and "indeterminate."

Now, we contend:

> The statements of a physical theory involve the second level only, each statement about individual systems being either empirical or derived from the second-level statements of physics.

Thus, if one has sufficient physical reasons for stating that a proposition a is true for some preparation x (second level of description), the property (or the properties) associated to a becomes an "actual" property; then, it can be deduced that any single system prepared in the same way certainly gives the answer "yes" if tested with any question belonging to a (first level of description), so that the aforesaid property can be attributed to any single system prepared according to x.

Analogous arguments hold if the proposition a is false; in this case a is associated to a property (or properties) which never can be attributed to any single system prepared according to x.

It must be noted that the predicate "false" defined in PR3.1 does not necessarily coincide with the usual logical predicate bearing the same name, which is defined as "not true." Indeed, "false" and "not true" coincide in classical physics, where, if the preparation x is perfectly known, *all* single systems prepared according to x must give the *same* answer (yes or no) if tested by a question α (i.e., no indeterminate question exists); but they do not coincide in quantum physics, where a question may be indeterminate notwithstanding a perfect knowledge of x.

The introduction of the predicate "false" here is motivated by physical arguments. Indeed, as we have seen above, the statement "a is false" allows well-defined predictions about single systems; furthermore, for every question α, a question α^{ν} exists whose truth implies the falsity of α. None of these properties hold for the predicate "not true."[7]

Finally, if a is "indeterminate," the property a cannot be attributed or denied for any single system prepared according to x.

3.4. Let us consider now the structures introduced in Section 2 on the set $\mathcal{Q}$ of questions and on the set $\mathcal{L}$ of propositions which are defined for some given physical system (see Section 3.2). On $\mathcal{Q}$ a unary operation $^{\nu}$ and an undetermined order functor $\prod$ are defined through PD2.3 and PD2.4, respectively. Furthermore, a quasiorder relation $<$, hence an equivalence relation $\sim$, is introduced on $\mathcal{Q}$ by means of D2.1, which is connected to $\prod$ by the following properties (where we briefly write $\alpha \prod \beta$ instead of $\prod_i \alpha_i$ whenever the family $\{\alpha_i\}$ reduces to a pair $\{\alpha, \beta\}$), whose proof is immediate.

$$\text{for every} \quad \alpha, \beta, \gamma \in \mathcal{Q} \qquad \begin{cases} \alpha < \beta \text{ iff } \alpha \sim \alpha \prod \beta \\ \alpha \prod \beta < \alpha \text{ and } \alpha \prod \beta < \beta \\ \gamma < \alpha \text{ and } \gamma < \beta \text{ imply } \gamma < \alpha \prod \beta \end{cases}$$

The crucial point here is that $<$ is a quasiorder relation, not a partial order (indeed, the antisymmetric property does not hold), so that $\prod$ is not a meet in the sense of lattice theory (note that the equivalence sign, not the equality sign, appears in the first condition above), nor is $^{\nu}$ an orthocomplementation. The latter statement becomes apparent if one observes that the following equivalence can be deduced from the second part of A3.3:

$$\text{for every} \quad \alpha, \beta \in \mathcal{Q}, \qquad \alpha \prod \beta \sim (\alpha^{\nu} \prod \beta^{\nu})^{\nu}$$

which shows that no de Morgan-type law links $\prod$ to $^{\nu}$.

[7]It is noteworthy that Popper's (1968) argument against the Birkhoff and Von Neumann (1936) thought-experiment intended to show that the nondistributivity of the propositional calculus of quantum mechanics rests on an equivocation between "false" and "not true" as defined here.

Thus, the structure $(\mathcal{Q}, \prod, {}^{\nu})$ is not an orthocomplemented lattice.

Let us come to the set of propositions $\mathcal{L} = \mathcal{Q}/_{\sim}$. Here, the quasiorder $<$ defined on $\mathcal{Q}$ induces a partial order $<$; then, PD2.1–PD2.4 (more correctly, as we have observed in Section 3.1, A3.1–A3.4) allow one to show that $(\mathcal{L}, <)$ is a lattice. Hence, two binary operators $\cup$ (join) and $\cap$ (meet) can be defined in $\mathcal{L}$ by means of standard mathematical techniques. By making use of axioms C and P a new operation can be introduced in $\mathcal{L}$, which proves to be an orthocomplementation, so that the "de Morgan laws" hold in $\mathcal{L}$:

$$\text{for every} \quad a, b \in \mathcal{L}, \qquad \begin{cases} a \cup b = (a' \cap b')' \\ a \cap b = (a' \cup b')' \end{cases}$$

Thus, the fundamental structure of the orthocomplemented lattice is obtained. This structure enjoys some further properties (it is weakly modular because of axioms C and P, and it is atomic and satisfies the covering law because of axiom A), which are relevant in the development of the theory.

3.5. Our discussion in Section 3.4 about $(\mathcal{Q}, \prod, {}^{\nu})$ and $(\mathcal{L}, \cup, \cap, ')$ shows that these structures are essentially different and, in particular, that the mapping

$$\varphi: \alpha \in \mathcal{Q} \mapsto a = [\alpha]_{\sim} \in \mathcal{L}$$

is not a homomorphism of $(\mathcal{Q}, \prod, {}^{\nu})$ onto $(\mathcal{L}, \cap, ')$. This must be stressed, since many arguments against the JP approach are based on an improper transfer of results from $(\mathcal{L}, \cap, ')$ to $(\mathcal{Q}, \prod, {}^{\nu})$ or conversely.

For instance, some criticisms of the JP approach are based on the belief that the set of the questions is an orthocomplemented lattice in the JP theory, which is false.

As another example, Mielinik's (1976) argument against the existence of an orthocomplementation on $\mathcal{L}$ is based on the belief that, for every $a \in \mathcal{L}$ and $\alpha \in \mathcal{Q}$ such that $\alpha \in a$, the equation $a' = [\alpha^{\nu}]_{\sim}$ holds in the JP approach, which is also false [it would be true if the mapping φ were an isomorphism; indeed, the equivalent equation $\varphi(\alpha)' = \varphi(\alpha^{\nu})$ would hold in this case].[8]

As we anticipated in the introduction, we do not intend to recognize and discuss all criticisms to the JP theory in the present paper; we limit ourselves here to the challenge by Hadjisavvas *et al.* (1980, 1981; Thieffine,

[8]In particular, Mielnik shows that a semitransparent mirror can be used in order to build up a question α_0 such that $\alpha_0 \sim \alpha_0^{\nu}$. Let us put $a_0 = [\alpha_0]_{\sim}$; then, the equation $a' = [\alpha^{\nu}]_{\sim}$ leads to $a_0 = a_0'$, which contradicts the statement that $'$ is an orthocomplementation on $\mathcal{L}$. Yet, this is not an objection to the JP theory (Garola, 1980) since the aforesaid equation is wrong. Indeed, for any α, $\beta \in \mathcal{Q}$ one has $\alpha \sim \beta$ whenever α is true iff β is true, while $\alpha^{\nu} \sim \beta^{\nu}$ whenever α is false iff β is false. Hence, one may have $\alpha \sim \beta$ while $\alpha^{\nu} \sim \beta^{\nu}$ does not hold, i.e.,

$$[\alpha]_{\sim} = [\beta]_{\sim} \qquad \text{while} \quad [\alpha^{\nu}]_{\sim} \neq [\beta^{\nu}]_{\sim}$$

1983), which will be positively solved in the present paper by means of a suitable mathematical model in Section 6 endowed with a well-established physical interpretation.

4. THE COMPLETELY FORMALIZED JP SYNTACTIC SYSTEM

4.1. As we anticipated in Section 3.1, the theoretical structure of any physical theory generally consists of (1) a mathematical structure and (2) an intended physical interpretation.

Our analysis in Section 3 (especially Sections 3.1 and 3.2) shows, in particular, that a more rigorous presentation of the basis of the JP theory would be desirable such that (i) the mathematical structure of the theory be clearly distinguished from its physical intended interpretation and from its possible mathematical models; (ii) all the axioms of the theory be plainly expressed and proved to be adequate for setting up the fundamental structures of the theory; (iii) a relative coherence proof be given.

We intend to provide such a presentation in this paper. More specifically, the present section will be devoted to the mathematical structure of the JP theory. It should be noted that our task might be performed by describing semiformally this mathematical structure by means of the usual mathematical language and of the logical-deductive structure of the common language. Alternatively, and more rigorously, a suitable formal language might be constructed in order to describe the mathematical language and the logical-deductive structure of the theory. Of course, the latter procedure is far less intuitive and requires rather cumbersome logical tools; furthermore, it leads to the construction of a conceptual instrument which is exceedingly powerful for our purposes. This notwithstanding, we will prefer it in the present section.

Our choice needs some justification. With reference to (i) above, we observe that a formal approach automatically discriminates the mathematical structure from its interpretation (in particular, the physical intended interpretation) and, within the former, the mathematical language from the logical-deductive structure expressed by means of the language itself; thus, it exemplifies a role that formal logic can play in the foundations of physics, it constitutes an interesting paradigm for further applications, and it illustrates the high levels of logical complexity which are inherent even in the primitive concepts of a physical theory.

With reference to (ii) above, we remark that a formal statement of any theory assures without further investigation that no unexplicit axiom is hidden within the ambiguities of the language (as could happen if the usual mathematical language were used). In our particular case, the axioms will result from our discussion in Section 3.2, i.e., they will be obtained by expressing formally A3.1–A3.4 and Piron's axioms C, P, A (we have already observed in Section 3.2 that this set of axioms turns out to be adequate). Of course, a full recognition of all the axioms supporting the theory, while it is a necessary step for proving its coherence, also is important for a better understanding of the theory itself.[9]

With reference to (iii) above, we note that a completely adequate formal presentation of any theory obliges one to classify the symbols that are introduced according to types [in the sense of Whitehead and Russell (1925)] and hence the well-formed formulas according to their orders. In our case, this shows at once that no proof of absolute coherence of the JP theory can be given, since the order of the formal language which is required when formalizing the theory is greater than one (Gödel).

On the other side, the formal presentation of all the specific axioms provides an unambiguous background for a proof of relative coherence by means of a mathematical model.

We would like to adjoin that the classification in types avoids any confusion between statements belonging to different language levels.[10] In our case, it shows that axioms A3.1–A3.4 must be distinguished from Piron's axioms C, P, A, since the former are first-order wffs, while the latter are second-order wffs.

A further argument in favor of a formal presentation of the theory may arise when considering that many objections against the JP approach concern the introduction of an inverse in the set of the questions and of an orthocomplementation in the complete lattice of the propositions. Then, a formal presentation of the JP theory shows that these operations are introduced at different language levels, and makes any confusion between them impossible (which may occur, in part, because of some ambiguities in the natural language by means of which the theory is usually discussed); moreover, it clearly exhibits the syntactical links between the operators $^{\nu}$ and $'$.

4.2. Following on our argument in Section 4.1, we will now introduce the mathematical structure of the JP approach to the foundations of quantum

[9] We stress that A3.1–A3.4 and C, P, A must be classified as specific axioms which belong to the mathematical language of the theory; they must be sharply distinguished from the logical axioms of the logical deductive structure, which will not be reported here. We recall that the logical status of the former is different from the status of the latter; for, the latter hold under any interpretation, while the former select the subclass of the possible interpretations of the theory. We also recall that the set of the logical axioms is incomplete, since the order of our formal language is greater than one (Gödel); hence, the whole set of all the axioms (both logical and specific) is necessarily incomplete.

[10] We will show in a forthcoming paper that the individual signs of type 0 in our formalized JP approach can be further interpreted as predicative signs of type 1 of an underlying physical language.

physics as a *completely formalized syntactic system* consisting of the following components.

a_1. A noninterpreted *formalized language*, in which primitive signs, derived signs, terms constructed by means of these signs, and well-formed formulas (wffs) are introduced.

a_2. *A logical deductive structure* (or *calculus*), in which some wffs of the formalized language are assumed as axioms, the set of the axioms being partitioned into the subset of the *logical axioms* and the subset of the *specific* (or *extralogical*) *axioms*, and some primitive inferential rules are introduced.

The basic formalized language **L** of the JP syntactic system will consist of a part (since we make use of monadic predicative variables only and introduce *ad hoc* restrictions in the formation rules for wffs) of a predicate calculus of the second order with identity, embodying the simple theory of types with partition of the 0-type terms into two kinds.

Following usual procedures, we shall construct **L** by giving an *alphabet*, i.e., a set of primitive signs classified according to syntactic categories and a (finite) set of *formation rules* (FR); this will be done by making use of a nonformalized metalanguage, consisting of a part of the English language together with some technical symbols, i.e., bold letters of the latin alphabet having the role of metalinguistic variables.

Then, the alphabet of **L** consists of the following signs.

Descriptive signs

D1. Individual signs of type 0 and kind $\mathscr{S}$: variables $x, y, z; x_1, y_1, z_1; \ldots; x_n, y_n, z_n; \ldots$.

D2. Individual signs of type 0 and kind $\mathscr{Q}$: variables $\alpha, \beta, \gamma; \alpha_1, \beta_1, \gamma_1; \ldots; \alpha_n, \beta_n, \gamma_n; \ldots$.

D3. Predicative signs of type 1: monadic variables ${}^1f, {}^1g, {}^1h; {}^1f_1, {}^1g_1, {}^1h_1; \ldots; {}^1f_n, {}^1g_n, {}^1h_n; \ldots$.

D4. Predicative signs of type 2: monadic variables ${}^2f, {}^2g, {}^2h, {}^2f_1, {}^2g_1, {}^2h_1; \ldots; {}^2f_n, {}^2g_n, {}^2h_n; \ldots$.

Specific signs

S1. Individual constant I of type 0 and kind $\mathscr{Q}$.

S2. One-argument functor $^{\nu}$.

S3. Undetermined order functor $\prod$.

S4. Diadic predicative constant T of type 1.

Logical signs

L1. Signs of connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

L2. Signs of quantifiers $\exists, \forall$.

L3. The identity sign $=$.

Auxiliary signs

AS. Comma and round parentheses.

The terms and the wffs of the language **L** are constructed by means of the following formation rules (FR)

FR for terms

T1. Every individual variable of type 0 and kind $\mathcal{S}$ or kind $\mathcal{Q}$ is a term of the same type and kind.

T2. The constant I is a term of type 0 and kind $\mathcal{Q}$.

T3. If $\boldsymbol{\alpha}$ is a term of type 0 and kind $\mathcal{Q}$, then $\boldsymbol{\alpha}^{\nu}$ is a term of type 0 and kind $\mathcal{Q}$.

T4. If **f** is a predicative (monadic) variable of type 1, then $\prod_{\mathbf{f}}$ (read: the product of all the α's that are **f**) is a term of type 0 and kind $\mathcal{Q}$.

We recall that a term where no free (i.e., not quantified) variable occurs is usually said to be a *closed* term, or *designator*; otherwise, it is said an *open* term (or *descriptive function*); in the present language there exists one closed primitive term only, the constant I.

FR for wffs

W1. For every term **x** of type 0 and kind $\mathcal{S}$ and term **t** of type 0 and kind $\mathcal{Q}$, $T(\mathbf{x}, \mathbf{t})$ is a wff.

W2. If **f** is a predicative variable, either of type 1 or of type 2, and **t** either is a term of type 0, kind $\mathcal{Q}$ or a predicative variable of type 1 respectively, then $\mathbf{f}(\mathbf{t})$ is a wff.

W3. If **s**, **t** are terms of the same type and kind, then $\mathbf{s} = \mathbf{t}$ is a wff.

W4. If **A**, **B** are wffs, then $\neg\mathbf{A}$, $\mathbf{A} \wedge \mathbf{B}$, $\mathbf{A} \vee \mathbf{B}$, $\mathbf{A} \rightarrow \mathbf{B}$, and $\mathbf{A} \leftrightarrow \mathbf{B}$ are wffs.

W5. If **A** is a wff, **t** an individual variable (of kind $\mathcal{S}$ or $\mathcal{Q}$), or a predicative variable of type 1, which occurs free in **A**, then $(\exists\mathbf{t})\mathbf{A}$, $(\forall\mathbf{t})\mathbf{A}$ are wffs.

A wff of **L** where no free variable occurs will be called a *closed* wff, or *sentence*; otherwise, it will be called an *open* wff, or *sentential* function (the FR for wff associate, in particular, atomic open wffs to every predicative variable or primitive predicate, and allow the introduction of derived predicates which are associated to molecular and/or general open wffs).

Furthermore, a (closed or open) wff or **L** will be called a *first-order* wff iff neither predicative variables of type 2 nor quantified predicative variables occur in it; otherwise, it will be called a *second-order* wff [thus, ${}^{1}f(\alpha)$, $T(x, \alpha)$, $(\forall\alpha)\,{}^{1}f(\alpha)$, $(\exists\alpha)\,{}^{1}f(\alpha)$, etc., are first-order wffs, while ${}^{2}f({}^{1}f)$, $(\forall\,{}^{1}f)\,{}^{2}f({}^{1}f)$, $(\exists\,{}^{1}f)\,{}^{2}f({}^{1}f)$, $(\exists\,{}^{1}f)({}^{1}f(\alpha))$, $(\forall\,{}^{1}f)({}^{1}f(\alpha))$, etc. are second-order wffs].

Our language **L** is thus constructed.

Now, we proceed to extend **L** in order to obtain a minimal class language $\mathbf{L}_e$ based on **L**, with the aid of the following signs.

Descriptive signs

D5. Class signs of type 1 and kind $\mathcal{Q}$: variables $a, b, c; a_1, b_1, c_1; \ldots; a_n, b_n, c_n; \ldots$.

Abstraction signs

A1. $\{\ldots\}$.

A2. The class abstraction operator $\{\cdot|\ldots\}$.

Logical sign

L4. The $\in$ operator.

By means of these signs, the following FR for class terms of kind $\mathcal{Q}$ are introduced.

- T5. Every class variable (necessarily of type 1, kind $\mathcal{Q}$) is a class term of type 1, kind $\mathcal{Q}$.
- T6. If $\mathbf{t}_1, \ldots, \mathbf{t}_n$ either are terms of type 0, kind $\mathcal{Q}$, then $\{\mathbf{t}_1, \ldots, \mathbf{t}_n\}$ is a finite class term of type 1 or 2, respectively, kind $\mathcal{Q}$.
- T7. If $\mathbf{A}(\mathbf{s})$ is a wff, $\mathbf{s}$ either is an individual or a class variable of kind $\mathcal{Q}$ which occurs free in it, then $\{\mathbf{s}|\mathbf{A}(\mathbf{s})\}$ is a class term of type 1 or 2, respectively, kind $\mathcal{Q}$.

The *substitutivity field* of the class variables is given by the set of all the class terms of type 1 constructed by means of T6 and T7.

Moreover, we give the following further FR for wffs.

- W6. If $\mathbf{s}$, $\mathbf{t}$ are class terms of the same type, then $\mathbf{s}=\mathbf{t}$ is a wff.
- W7. If $\mathbf{t}$ is a class term, either of type 1 or of type 2, and $\mathbf{s}$ either is a term of type 0, kind $\mathcal{Q}$, or a class term of type 1 respectively, then $\mathbf{s}\in\mathbf{t}$ is a wff.
- W8. If $\mathbf{A}$ is a wff, $\mathbf{t}$ a class variable which occurs free in $\mathbf{A}$, then $(\exists\mathbf{t})\mathbf{A}$ and $(\forall\mathbf{t})\mathbf{A}$ are wffs.

Then, we denote by $\mathbf{L}_e$ the language whose alphabet consists of the descriptive signs D1–D5, of the specific signs S1–S4, of the logical signs L1–L4, of the abstraction signs A1 and A2, of the auxiliary signs AS, whose terms are constructed by means of the FR T1–T7, and whose wffs are constructed by means of the FR W1–W8.

The above definitions of closed and open terms and wffs, and of first-order and second-order wffs, suitably completed by introducing class variables, are maintained in $\mathbf{L}_e$. Furthermore, we assume that all the usual logical axioms and the inference rules of the classical second-order predicate calculus including class logic hold in $\mathbf{L}_e$.

The following identities and equivalences, which can be introduced following standard procedures (Whitehead and Russell, 1925) allow the reduction of wffs of the enlarged language $\mathbf{L}_e$ into wffs of $\mathbf{L}$.

R_1. If $\mathbf{t}$ is a class term, either of type 1 or of type 2, and $\mathbf{r}$ either is an individual variable of kind $\mathcal{Q}$ or a class variable respectively, then the sentential function $\mathbf{r} \in \mathbf{t}$ is determining, i.e., such that

$$\mathbf{t} = \{\mathbf{r} \mid \mathbf{r} \in \mathbf{t}\}$$

Furthermore, let $\mathbf{s}$ be a term of kind $\mathcal{Q}$ and type 0 if $\mathbf{t}$ is of type 1, type 1 if $\mathbf{t}$ is of type 2. Whenever $\mathbf{t} = \{\mathbf{t}_1, \ldots, \mathbf{t}_n\}$, i.e., $\mathbf{t}$ is a finite class term, then

$$\mathbf{s} \in \mathbf{t} \leftrightarrow ((\mathbf{s} = \mathbf{t}_1) \vee \ldots \vee (\mathbf{s} = \mathbf{t}_n))$$

Whenever $\mathbf{t} = \{\mathbf{r} | \mathbf{A}(\mathbf{r})\}$, with $\mathbf{r}$ a variable $\mathbf{A}(\mathbf{r})$ a wff of $\mathbf{L}_e$, then

$$\mathbf{s} \in \mathbf{t} \leftrightarrow \mathbf{A}(\mathbf{s})$$

R_2. If $\mathbf{s}$ and $\mathbf{t}$ are class terms, either of type 1 or of type 2, and $\mathbf{r}$ either is an individual variable of kind $\mathcal{Q}$ or a class variable respectively, then

$$\mathbf{s} = \mathbf{t} \leftrightarrow (\forall \mathbf{r})((\mathbf{r} \in \mathbf{s}) \leftrightarrow (\mathbf{r} \in \mathbf{t}))$$

R_3. For every class term $\mathbf{t}$ of type 1 of $\mathbf{L}_e$ (hence $\mathbf{t} = \{\boldsymbol{\alpha} | \boldsymbol{\alpha} \in \mathbf{t}\}$), a primitive derived predicate $\mathbf{g}$ exists such that

$$\mathbf{t} = \{\boldsymbol{\alpha} | \mathbf{g}(\boldsymbol{\alpha})\}$$

(hence, for every term s of type 0, kind $\mathcal{Q}$, $\mathbf{s} \in \mathbf{t} \leftrightarrow \mathbf{g}(\mathbf{s})$). In particular, for any class variable $\mathbf{a}$, $\mathbf{a} = \{\boldsymbol{\alpha} | \mathbf{f}(\boldsymbol{\alpha})\}$, with $\mathbf{f}$ a free predicative variable of type 1.

R_4. For every class term $\mathbf{t}$ of type 2 of $\mathbf{L}_e$, a second-order sentential function $\mathbf{A}(\mathbf{f})$ that belongs to $\mathbf{L}$ exists, with $\mathbf{f}$ a free predicative variable of type 1, such that, for any class term $\mathbf{s} = \{\boldsymbol{\alpha} | \mathbf{g}(\boldsymbol{\alpha})\}$ of type 1

$$\mathbf{s} \in \mathbf{t} \leftrightarrow \mathbf{A}(\mathbf{g})$$

The rule R_1 states that every class term can be constructed by means of a sentential function of $\mathbf{L}_e$; furthermore, because of R_3, R_4, every wff of $\mathbf{L}_e$ can be translated into a wff of $\mathbf{L}$; hence, the wffs of the enlarged language $\mathbf{L}_e$ can be reduced into wffs of $\mathbf{L}$. Thus, our extension of $\mathbf{L}$ into $\mathbf{L}_e$ is not logically needed, even if it greatly simplifies our treatment.

Finally, we convene that in the term

$$\prod_{\mathbf{f}}$$

the predicate $\mathbf{f}$ may be substituted by the 1-type class term $\mathbf{t} = \{\boldsymbol{\alpha} | \mathbf{f}(\boldsymbol{\alpha})\}$ (hence by a class variable).

Then we call *formalized question preparation structure*, or fqps for short, the calculus constructed in $\mathbf{L}_e$ by adjoining to the logical axioms, as specific axioms, the following (closed) wffs.

Axiom 1. $(\forall x)T(x, I)$.

Axiom 2. $(\forall \alpha)(\alpha^{\nu\nu} = \alpha)$.

Axiom 3. $$(\forall a)(\forall x)\left(\left(((\forall \alpha)((\alpha \in a) \to T(x, \alpha))) \leftrightarrow T\left(x, \prod_a\right)\right)\right.$$
$$\left.\wedge \left(((\forall \alpha)((\alpha \in a) \to T(x, \alpha^\nu))) \leftrightarrow T\left(x, \left(\prod_\alpha\right)^\nu\right)\right)\right).$$

Axiom 4. $(\forall \alpha)(\forall x)(\neg(T(x, \alpha) \wedge T(x, \alpha^\nu)))$.

In any fqps we define the derived signs $<$, $\sim$, $[\,\cdot\,]_\sim$, ${}^2T$, $\mathscr{L}$, $<$ by means of the following (open) definitions, respectively.

D4.1. $\alpha < \beta \leftrightarrow (\forall x)(T(x, \alpha) \to T(x, \beta))$.
D4.2. $\alpha \sim \beta \leftrightarrow (\alpha < \beta) \wedge (\beta < \alpha)$.
D4.3. $[\alpha]_\sim = \{\beta | \beta \sim \alpha\}$.
D4.4. ${}^2T(x, a) \leftrightarrow (\forall \alpha)((\alpha \in a) \to T(x, \alpha))$.
D4.5. $\mathscr{L} = \{a | (\exists \alpha)(a = [\alpha]_\sim)\}$.
D4.6. $a < b \leftrightarrow (\forall x)({}^2T(x, a) \to {}^2T(x, b))$.

By making use of classical inference rules and axioms together with the above definitions, one can formally prove the following statements.

P4.1. (i) $(\forall \alpha)(\alpha < \alpha)$.
(ii) $(\forall \alpha)(\forall \beta)(\forall \gamma)(((\alpha < \beta) \wedge (\beta < \gamma)) \to (\alpha < \gamma))$.
P4.2. (i) $(\forall \alpha)(\alpha \sim \alpha)$.
(ii) $(\forall \alpha)(\forall \beta)((\alpha \sim \beta) \leftrightarrow (\beta \sim \alpha))$.
(iii) $(\forall \alpha)(\forall \beta)(\forall \gamma)(((\alpha \sim \beta) \wedge (\beta \sim \gamma)) \to (\alpha \sim \gamma))$.
P4.3. (i) $(\forall a)(a < a)$.
(ii) $(\forall a)(\forall b)(((a < b) \wedge (b < a)) \leftrightarrow (a = b))$.
(iii) $(\forall a)(\forall b)(\forall c)(((a < b) \wedge (b < c)) \to (a < c))$.

Furthermore, the fundamental theorem for fqps holds (Piron, 1976a), the formal proof of which is reported in the Appendix.

Theorem 4.1:
(i) $(\forall a)(\exists b)((b \in \mathscr{L}) \wedge ((\forall \alpha)((\alpha \in a) \to (b < [\alpha]_\sim)))$
$\wedge ((\forall c)(((c \in \mathscr{L}) \wedge ((\forall \alpha)((\alpha \in a) \to (c < [\alpha]_\sim)))) \to (c < b))))$.
(ii) $(\forall a)(\exists b)((b \in \mathscr{L}) \wedge ((\forall \alpha)((\alpha \in a) \to ([\alpha]_\sim < b)))$
$\wedge ((\forall c)(((c \in \mathscr{L}) \wedge ((\forall \alpha)((\alpha \in a) \to ([\alpha]_\sim < c)))) \to (b < c))))$.

We intend now to define a stronger structure. With this aim in mind we introduce the following further definitions:

D4.7. (i) $(a \cap b = c) \leftrightarrow ((a \in \mathscr{L}) \wedge (b \in \mathscr{L}) \wedge (c \in \mathscr{L}) \wedge ((\forall x)({}^2T(x, c) \leftrightarrow ({}^2T(x, a) \wedge {}^2T(x, b)))))$.

(ii) $(a \cup b = c) \leftrightarrow ((a \in \mathscr{L}) \wedge (b \in \mathscr{L}) \wedge (c \in \mathscr{L}) \wedge ((\forall x)({}^2T(x, c) \leftrightarrow ({}^2T(x, a) \vee {}^2T(x, b)))))$.

D4.8. $O = I^{\nu}$.

D4.9. $(\mathbf{1} = [I]_{\sim}) \wedge (\mathbf{0} = [O]_{\sim})$.

D4.10. $(b \in \{a'\}) \leftrightarrow ((a \in \mathscr{L}) \wedge (b \in \mathscr{L}) \wedge ((\exists \alpha)((\alpha \in a) \wedge (\alpha^{\nu} \in b))) \wedge ((a \cap b) = \mathbf{0}) \wedge ((a \cup b) = \mathbf{1}))$.

D4.11. (i) $(b \mathscr{C} a) \leftrightarrow (((a \in \mathscr{L}) \wedge (b \in \mathscr{L}) \wedge (\neg(a = b)) \wedge (a < b) \wedge ((\forall c)((c \in \mathscr{L}) \wedge (a < c) \wedge (c < b)))) \rightarrow ((c = a) \vee (c = b)))$

(ii) $a \in \mathscr{A} \leftrightarrow a \mathscr{C} \mathbf{0}$

Then, we call JP *formalized question-preparation structure*, or JP fqps, the calculus constructed in $\mathbf{L}_e$ by selecting axioms 1–4 together with the following wffs.

Axiom C. $(\forall a)(a \in \mathscr{L}) \rightarrow ((\exists b)(b \in \{a'\}))$.

Axiom P. $(\forall a)(\forall b)((a < b) \leftrightarrow ((\forall c)(\forall d)(((c \in \{a'\}) \wedge (d \in \{b'\})) \rightarrow (d < c))))$.

Axiom A.

A_1. $(\forall a)(((a \in \mathscr{L}) \wedge (\neg(\alpha = 0))) \rightarrow ((\exists b)((b \in \mathscr{A}) \wedge (b < a))))$.

A_2. $(\forall a)(\forall b)(((a \in \mathscr{L}) \wedge (b \in \mathscr{A}) \wedge ((a \cap b) = 0)) \rightarrow ((a \cup b) \mathscr{C} a))$.

Coherently with our definitions at the beginning of the present section, we call *completely formalized JP syntactic system* the pair made up by the language $\mathbf{L}_e$ and the JP fqps.

Finally, by making use of classical inference rules and axioms together with the above axioms C and P and definitions D3.8–D3.11, one can formally prove the following basic statements for JP fqps.

P4.4. (*Orthocomplementation*)

$(\forall a)(\forall b)(\forall c)(((b \in \{a'\}) \wedge (c \in \{a'\})) \rightarrow (b = c))$.

$(\forall a)(\forall b)(\forall c)(((b \in \{a'\}) \wedge (c \in \{b'\})) \rightarrow (c = a))$.

$(\forall a)(\forall b)((b \in \{a'\}) \rightarrow (((a \cap b) = 0) \wedge ((a \cup b) = \mathbf{1})))$.

P4.5. (*Weak modularity*)

$(\forall a)(\forall b)(\forall c)(((a \in \mathscr{L}) \wedge (b \in \mathscr{L}) \wedge (a < b) \wedge (c \in \{b'\}) \wedge (d = (a \cup c))) \rightarrow ((b \cap d) = a))$.

4.3. Now we make some technical comments on our formalization above, mainly concerning our language $\mathbf{L}_e$.

First, we observe that some general language exists in classical logic, i.e., predicate logic of higher order with identity embodying the (simple) theory of types and a partition of the 0-type terms in kinds (Carnap, 1958), which could have been used in order to formalize the JP theory. Our language $\mathbf{L}_e$ here is an *ad hoc* language, which has been endowed with only those tools required for our limited purposes (of course, the formalization of the whole mathematical structure of the JP theory lies beyond the aims of the present paper; for, it would require a number of further definitions and cumbersome formal proofs, while it would not improve our understanding of the basic concepts), favoring manageability and immediate interpretability at the expense of generality and elegance. As a consequence of this choice, the number of symbols that have been introduced in $\mathbf{L}_e$ is not minimal at all. For instance, all the usual signs of logical connectives have been inserted in the alphabet, while they could be interdefined. Analogously, D4.3 and D4.5 have been respectively chosen in place of the equivalent (because of R_1–R_4) definitions

D4.3′. $\beta \in [\alpha]_{\sim} \leftrightarrow \beta \sim \alpha$
D4.5′. $(a \in \mathscr{L}) \leftrightarrow (\exists \alpha)((\forall \beta)((\beta \in a) \leftrightarrow (\beta \in [\alpha]_{\sim})))$

which would have dispensed with the class abstraction operator and the identity sign.

Second, let us recall that we have shown in Section 3.3 that an explicit introduction of some new predicates (*false*, *indeterminate*) and concepts (*preparation*, *repeated measurements*) would have clarified (even from an epistemological point of view) the basic notions of the JP approach. Then, we remark that the language $\mathbf{L}_e$ may be further enlarged by introducing two derived predicates F and U by means of the following (open) definitions.

$$F(x, \alpha) \leftrightarrow T(x, \alpha^{\nu}).$$
$$U(x, \alpha) \leftrightarrow (\neg(T(x, \alpha) \vee F(x, \alpha))).$$

By making use of F, the Axioms 3 and 4 in Sections 4.2 can be respectively restated as follows.

Axiom 3′. $$(\forall a)(\forall x)\left(\left(((\forall \alpha)((\alpha \in a) \rightarrow T(x, \alpha))) \leftrightarrow T\left(x, \prod_a\right)\right) \wedge \left(((\forall \alpha)((\alpha \in a) \rightarrow F(x, \alpha))) \leftrightarrow F\left(x, \prod_a\right)\right)\right)$$

Axiom 4′. $(\forall \alpha)(\forall x)(\neg(T(x, \alpha) \wedge F(x, \alpha)))$.

Third, we notice that $\mathbf{L}_e$ formalizes the language used by Piron when talking about *questions* and *propositions*; hence, in some sense, it could be said to be a metalanguage with respect to the language of the physical properties, which correspond to the propositions. In this metalanguage, two levels can be distinguished (not to be confused with the levels of description introduced in Section 3.3), the first regarding questions, the second one *classes* of *questions* (in particular, *propositions*). When formalizing, the second level must be reached right from the beginning in order to define properly Piron's functor $\prod$, whose order, according to Piron, may be infinite. Yet, Piron's definition of $\prod$ introduces an unphysical element in the theory that one could prefer to avoid, defining $\prod$ only for a finite (though indeterminate in number) sequences of questions.

Thus, a language $\mathbf{L}_w$ weaker than L_e can be introduced by substituting T4 for terms with the following statement.

T4′. If $\mathbf{t}$ is a finite class term of type 1 (hence of kind $\mathcal{Q}$), then $\prod_{\mathbf{t}}$ is a term of type 0 and kind $\mathcal{Q}$.

Let us call weak *formalized question proposition structure*, or wfqps, the calculus constructed in $\mathbf{L}_w$ by selecting the axioms 1, 2, 4 above together with the weaker form of axiom 3 obtained by restricting a in $\prod_a$ to be a finite class term, or equivalently,

Axiom 3_w.

$$(\forall\alpha_1)\ldots(\forall\alpha_n)(\forall x)\Bigg(\Bigg(T\Bigg(x, \prod_{\{\alpha_1,\ldots,\alpha_n\}}\Bigg)\leftrightarrow(T(x,\alpha_1)\wedge\ldots\wedge T(x,\alpha_n))\Bigg)$$
$$\wedge\Bigg(F\Bigg(x, \prod_{\{\alpha_1,\ldots,\alpha_n\}}\Bigg)\leftrightarrow(F(x,\alpha_1)\wedge\ldots\wedge F(x,\alpha_n))\Bigg)\Bigg)$$

Axiom 3_w might be easily formulated without making use of the class theory and second-order predicate logic; hence, all the fundamental axioms of a wfqps can be formulated by making use of the first-order predicate logic only.

Yet, it must be noted that in $\mathbf{L}_w$ it is impossible to prove that the partially ordered set of the propositions is a lattice. More specifically, even if the statements (i) and (ii) in Theorem 4.1 are substituted by the weaker statements

(i′) $(\forall b)(\forall c)(((b\in\mathcal{L})\wedge(c\in\mathcal{L}))$
$\rightarrow((\exists d)((d\in\mathcal{L})\wedge((\forall x)({}^2T(x,d)\leftrightarrow({}^2T(x,b)\wedge{}^2T(x,c))))))$

(ii′) $(\forall b)(\forall c)(((b\in\mathcal{L})\wedge(c\in\mathcal{L}))$
$\rightarrow((\exists d)((d\in\mathcal{L})\wedge((\forall x)({}^2T(x,d)\leftrightarrow({}^2T(x,b)\vee{}^2T(x,c))))))$

which hold in $\mathbf{L}_e$ whenever the class term a is bound to coincide with a two-element (hence finite) class term $\{\beta,\gamma\}$, the statement (ii′) cannot be proved in $\mathbf{L}_w$, since its proof requires full use of axiom 3 in its original form.

Finally, we stress that the wff $(\alpha\sim\beta)\leftrightarrow(\alpha^\nu\sim\beta^\nu)$ is *not* a theorem in the fqps; hence, neither $([\alpha]_\sim=[\beta]_\sim)\leftrightarrow([\alpha^\nu]_\sim=[\beta^\nu]_\sim)$ is a theorem. This suffices to invalidate the basic assumptions of some criticism to Piron's approach (Mielnik, 1976) that we have already discussed in Section 3.

5. THE PHYSICAL INTENDED INTERPRETATION OF THE COMPLETELY FORMALIZED JP SYNTACTIC SYSTEM

The completely formalized JP syntactic system introduced in Section 4.2 will be endowed of an intended physical interpretation by means of the following interpretation rules.

1. Every individual variable of kind $\mathscr{S}$ ranges over the set of the preparation procedures (which we denote again by $\mathscr{S}$, by abuse of language), implicitly introduced in PR2.1.
2. Every individual variable of kind $\mathscr{Q}$ ranges over the set of the questions (which we denote again by $\mathscr{Q}$, by abuse of language), defined according to PD2.1.
3. Every predicative monadic variable (of type 1 or 2) ranges over a set of physical properties (of type 1 or 2) of the questions (it must be clearly understood that these properties of the questions must *not* be confused with the properties of the physical system introduced in D2.4).
4. The individual constant I is associated to the trivial question, defined according to PD2.2.
5. The functorial sign $^{\nu}$ is associated to the mapping

$$^{\nu}\colon \alpha \in \mathscr{Q} \mapsto \alpha^{\nu} \in \mathscr{Q}$$

   α^{ν} being the inverse of α, defined according to PD2.3.
6. The functorial sign $\prod$ is associated to the mapping

$$\prod\colon a = \{\alpha_i\} \in \mathscr{P}(\mathscr{Q}) \mapsto \prod_a = \prod_i \alpha_i \in \mathscr{Q}$$

   where $\{\alpha_i\}$ is any family of questions, $\mathscr{P}(\mathscr{Q})$ is the power set of $\mathscr{Q}$, and $\prod_i \alpha_i$ is defined according to PD2.4.
7. The diadic predicate T is associated to the "true" predicate, defined according to PR2.1.
8. The signs $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\exists$, $\forall$, $=$ are intrepreted according to the usual logical conventions.

Then, the following further interpretation rules will be given for the new signs introduced in L_e.

9. Every class variable ranges over $\mathscr{P}(\mathscr{Q})$.
10. The signs $\{\ldots\}$, $\{\cdot\,|\ldots\}$, $\in$ are interpreted according to the usual conventions in class theory [hence, every class term of type m is interpreted over $\mathscr{P}^m(\mathscr{Q})$].

Finally, the following derived interpretation rule applies to the predicates introduced in Section 4.3.

11. The auxiliary predicates F and U are associated to the "false" and "indeterminate" predicates, defined according to PR3.1 and PR3.2, respectively.

According to the above interpretation rules, the axioms 1–4 in Section 4.2 formally state A3.1–A3.4 in Section 3.2. The equivalent forms 3′ and 4′ of the axioms 3 and 4 given in Section 4.3 respectively formalize A3.3′ and A3.4′ in Section 3.3.

Then, the definitions D4.1–D4.6 respectively formalize JP definitions D2.1–D2.6. Furthermore, the propositions P4.1–P4.3 show that $<$, $\sim$, and $<$ can respectively be considered a quasiorder relation on the set of the questions, an equivalence relation on the same set, and a partial order relation on the set $\mathscr{L} = \mathscr{Q}/_{\sim}$ of the propositions.

Theorem 4.1 formally states the fundamental Theorem 2.1 of the JP theory [the statements (i′) and (ii′) of Section 4.3 being equivalent to the statement that the poset $\mathscr{L}$ of the propositions is a lattice]. The definitions D4.7, D4.10, and D4.11 formally introduce the symbols $\cap$ and $\cup$ in $\mathscr{L}$, the "compatible complement" in the sense of Piron, a "covering" relation in $\mathscr{L}$, and the concept of "atom," respectively. Then, axioms C and A formalize the axioms bearing the same name in Piron (1976a), while axiom P here formalizes a statement which is equivalent to axiom P in the Piron approach (whose formal statement would have been rather cumbersome). Finally, P4.4 and P4.5 formalize well-known basic properties of the lattice $\mathscr{L}$ following from axioms C, P.

6. THE (RELATIVE) COHERENCE OF THE JP APPROACH

6.1. We have seen in Section 3.2 and 4.2 that the JP theory does not stand upon Piron's axioms C, P, A only, but also upon axioms A3.1–A3.4. Thus, a coherence problem arises for the whole set of axioms, which is not solved by the existence of a physical model, like that in Section 5, nor by a formalized presentation of the mathematical structure of the theory, like the one proposed in Section 4.2, though the latter provides a suitable background for the discussion of the problem.

We have already observed in Section 4.1 that only relative, not absolute coherence can be proved, because of the Gödel theorems; thus, all that can be done is to show that the theory is coherent if mathematics is such. As usual, this may be proved by means of a suitable mathematical model where all the primitive symbols and operators of the theory are suitably represented, and all the axioms are logically true. Now, there exist models of the JP theory in the literature which can be easily adapted to our present framework in order to attain a (relative) coherence proof; in particular, the infinitary model exhibited in Hadjisavvas *et al.* (1980) and a finitary model displayed in Foulis and Randall (1984).[11] Yet, we would like to accept a challenge issued in Hadjisavvas *et al.* (1980) [and renewed in Hadjisavvas and Thieffine (1984)]; thus, we will prove in the next subsection the coherence of the axioms A3.1–A3.4 and C, P, A by means of a model which, besides being formally adequate, is such that every basic mathematical object and relation in it can be physically interpreted.

In order to achieve a better understanding of this point, let us recall that a physical question corresponds to a measuring apparatus (see Section 2) but not generally to an observable according to the usual QP meaning [i.e., in the sense formalized, e.g., by Mackey (1963)].

Whenever a question α satisfies some probability requirements (Garola, 1985) it belongs to some suitable equivalence class of questions, or *effect*, which, according to a recent approach to QP (e.g., Ludwig, 1983; Garola and Solombrino, 1983), can be represented by means of a linear, bounded, self-adjoint operator α on a Hilbert space, such that $0 \leq \alpha \leq 1$ (the states being represented, as usual, by suitable trace class operators).

Hence, this representation yields a many-to-one representation of questions onto suitable operators which is embodied in our mathematical model; indeed, according to the latter, every variable of $\mathbf{L}_e$ which is bound to range over questions in the physical intended interpretation is made to range over operators representing questions, according to the aforesaid representation, in our model. It must be stressed that the spectral values of the representative operators must not be interpreted as possible outcomes of measurements of the corresponding questions, but as probabilities of the yes outcome (Garola and Solombrino, 1983); therefore, a question must not be confused with the observable represented by the same operator according to the usual Hilbert representation. We also remark that every variable of $\mathbf{L}_e$ which is bound to range over states in the physical intended interpretation is made to range over operators representing pure states (according to the aforesaid representation) in our model.

Thus, our model is endowed with a well-defined physical meaning.

6.2. Let $\mathscr{H}$ be a Hilbert space. We denote by $\mathscr{F}(\mathscr{H})$ the set of all the linear, bounded, self-adjoint operators α on $\mathscr{H}$ such that $0 \leq \alpha \leq 1$, and by $\mathscr{E}(\mathscr{H}) \subseteq \mathscr{F}(\mathscr{H})$ the set of all the (orthogonal) projection operators on $\mathscr{H}$. Furthermore, we denote by $\circ$ the composition of mappings, and for every

[11] We emphasize that our aim cannot be reached by means of the Hilbert realization discussed in Piron (1976a), which is an alternative formulation of the usual Hilbert space model of QM. Indeed, the latter provides a mathematical representation of derived symbols (propositions) and operations (orthocomplementation, join, meet) only, and does not prove the (relative) coherence of the whole set of the axioms of the JP theory, which also contains the axioms that Piron did not make explicit, and axiom C involving both questions and propositions.

$\alpha \in \mathscr{F}(\mathscr{H})$ we call E_α the resolution of the identity that belongs to α; finally, for every trace class operator ρ we denote by $\mathrm{Tr}[\rho]$ the trace of ρ.

Then, we assume that the following correspondence rules hold.

1. Every individual variable of kind $\mathscr{S}$ ranges over the subset $\mathscr{S}(\mathscr{H}) \subseteq \mathscr{E}(\mathscr{H})$ of all the projections over one-dimensional subspaces of $\mathscr{H}$.
2. Every individual variable of kind $\mathscr{Q}$ ranges over $\mathscr{F}(\mathscr{H})$.
3. Every predicative monadic variable (of type 1 or 2) ranges over a set of mathematical properties (of type 1 or 2) of the operators in $\mathscr{F}(\mathscr{H})$.
4. The individual constant I denotes the identity operator on $\mathscr{H}$.
5. The functorial sign $^{\nu}$ denotes the mapping

$$^{\nu}\colon \alpha \in \mathscr{F}(\mathscr{H}) \mapsto \alpha^{\nu} = (I - \alpha) \in \mathscr{F}(\mathscr{H})$$

6. The functorial sign $\prod$ denotes the mapping

$$\prod\colon \quad a \in \mathscr{P}(\mathscr{F}(\mathscr{H})) \mapsto \prod_a = \frac{1}{2}\left(\bigcap_{\alpha \in a} E_\alpha(\{1\}) + \left(\bigcap_{\alpha \in a} E_\alpha(\{0\})\right)^{\nu}\right) \in \mathscr{F}(\mathscr{H})$$

[equivalently

$$\prod\colon \quad \{\alpha_i\} \mapsto \frac{1}{2}\left(\bigcap_i E_{\alpha_i}(\{1\}) + \left(\bigcap_i E_{\alpha_i}(\{0\})\right)^{\nu}\right)$$

$\{\alpha_i\}$ being any family of operators of $\mathscr{F}(\mathscr{H})$] with $\mathscr{P}(\mathscr{F}(\mathscr{H}))$ the power set of $\mathscr{F}(\mathscr{H})$ and $\bigcap_{\alpha \in a}$ the usual meet in the lattice of all the (orthogonal) projection operators on $\mathscr{H}$ [we recall that the greatest lower bound $P \cap Q$ of two projections P and Q is the projection onto the intersection of their ranges, while the least upper bound $P \cup Q$ is the projection onto the closure of the sum of their ranges; furthermore we explicitly note that generally $\prod_a$ is not a projection, even when $a \subseteq \mathscr{E}(\mathscr{H})$].

7. The diadic predicate T denotes the (mathematical) property "the trace is 1"; more precisely, for every $x \in \mathscr{S}(\mathscr{H})$, $\alpha \in \mathscr{F}(\mathscr{H})$,

$$T(x, \alpha) \quad \text{iff} \quad \mathrm{Tr}[x \circ \alpha] = 1$$

(equivalently, $\mathrm{Tr}[x \circ E_\alpha(\{1\})] = 1$).

8. The signs $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\exists$, $\forall$, $=$, comma, and round parentheses are interpreted according to the usual logical conventions.

Then, let the following further correspondence rules hold for the new signs introduced in the extended language $\mathbf{L}_e$.

9. Every class variable ranges over $\mathscr{P}(\mathscr{F}(\mathscr{H}))$.
10. The signs $\{\ldots\}$, $\{\cdot \mid \ldots\}$, and $\in$ are interpreted according to the usual conventions in class theory (hence, every class term of type m is interpreted over $\mathscr{P}^m(\mathscr{F}(\mathscr{H}))$).

Finally, the following correspondence rule applies to the predicates introduced in Section 4.3.

11. The auxiliary predicates F and U denote the (mathematical) properties "the trace is 0" and "the trace is more than 0 and less than 1", respectively; more precisely, for every $x \in \mathcal{S}(\mathcal{H})$, $\alpha \in \mathcal{F}(\mathcal{H})$,

$$F(x, \alpha) \quad \text{iff} \quad \mathrm{Tr}[x \circ \alpha] = 0$$

$$U(x, \alpha) \quad \text{iff} \quad 0 < \mathrm{Tr}[x \circ \alpha] < 1$$

According to the above rules of correspondence, axioms 1–4 in Section 4.2 formally state the following mathematical propositions (whose proof is straightforward).

P6.1. Let $x \in \mathcal{S}(\mathcal{H})$; then $\mathrm{Tr}[x \circ I] = \mathbf{1}$.
P6.2. For every $\alpha \in \mathcal{F}(\mathcal{H})$, $\alpha^{\nu\nu} = \alpha$.
P6.3. Let $x \in \mathcal{S}(\mathcal{H})$, $a \in \mathcal{P}(\mathcal{F}(\mathcal{H}))$; then,

$\mathrm{Tr}[x \circ \alpha] = 1$ for every $\alpha \in a$

$$\text{iff} \quad \mathrm{Tr}\left[x \circ \left(\frac{1}{2} \bigcap_{\alpha \in a} E_\alpha(\{1\}) + \frac{1}{2}\left(\bigcap_{\alpha \in a} E_\alpha(\{0\})\right)^{\nu}\right)\right] = 1$$

$\mathrm{Tr}[x \circ \alpha] = 0$ for every $\alpha \in a$

$$\text{iff} \quad \mathrm{Tr}\left[x \circ \left(\frac{1}{2} \bigcap_{\alpha \in a} E_\alpha(\{1\}) + \frac{1}{2}\left(\bigcap_{\alpha \in a} E_\alpha(\{0\})\right)^{\nu}\right)\right] = 0$$

P6.4. Let $x \in \mathcal{S}(\mathcal{H})$; then the properties $\mathrm{Tr}[x \circ \alpha] = 1$ and $\mathrm{Tr}[x \circ \alpha^{\nu}] = 1$ are mutually exclusive.

Furthermore, the definitions D4.1–D4.6 in Section 4.2 can be translated into the following definitions respectively.

D6.1. $\alpha < \beta$ iff $E_\alpha(\{1\}) < E_\beta(\{1\})$.
D6.2. $\alpha \sim \beta$ iff $E_\alpha(\{1\}) = E_\beta(\{1\})$.
D6.3. $[\alpha]_\sim = \{\beta | E_\beta(\{1\}) = E_\alpha(\{1\})\}$.
D6.4. ${}^2T(x, a)$ iff $\mathrm{Tr}[x \circ \alpha] = 1$ for every $\alpha \in a$.
D6.5. $\mathcal{L}(\mathcal{H}) = \{a | a = [\alpha]_\sim$ for some $\alpha \in \mathcal{F}(\mathcal{H})\}$.
D6.6. $a < b$ iff $E_\alpha(\{1\}) < E_\beta(\{1\})$ for every $\alpha \in a$, $\beta \in b$.

Trivially, $<$ denotes a quasiorder relation (which does not coincide with the usual ordering) on $\mathcal{F}(\mathcal{H})$, $\sim$ an equivalence relation on the same set, and $<$ an order relation on $\mathcal{L}(\mathcal{H})$.

In Section 4.2, Theorem 4.1 formally states that $(\mathcal{L}(\mathcal{H}), <)$ is a complete lattice. In our present model, the same result can be obtained by observing that the mapping

$$\phi: \quad a = [\alpha]_\sim \in \mathcal{L}(\mathcal{H}) \mapsto E_\alpha(\{1\}) \in \mathcal{E}(\mathcal{H})$$

is well defined, and is an order isomorphism of $(\mathscr{L}(\mathscr{H}), <)$ onto the complete lattice $(\mathscr{E}(\mathscr{H}), <)$ [it must be noted that the restriction to $\mathscr{E}(\mathscr{H})$ of the quasiorder $<$ defined on $\mathscr{F}(\mathscr{H})$ is a partial order relation which coincides with the usual ordering on $\mathscr{E}(\mathscr{H})$].

Then, D4.7–D4.11 can be translated into the following definitions respectively.

D6.7. Let $a, b \in \mathscr{L}(\mathscr{H})$, $\alpha \in a$, $\beta \in b$; then $\cap$ and $\cup$ denote meet and join in $\mathscr{L}(\mathscr{H})$, respectively [hence, if we also denote by $\cap$ and $\cup$ the meet and join in $\mathscr{E}(\mathscr{H})$, respectively, then $a \cap b = \phi^{-1}(\phi(a) \cap \phi(b))$ and $a \cup b = \phi^{-1}(\phi(a) \cup \phi(b))$].

D6.8. We denote by O the null operator.

D6.9. We denote by **1** and **0**, respectively, the equivalence classes of I and O with respect to $\sim$.

D6.10. Let $a \in \mathscr{L}(\mathscr{H})$. Then, $b \in \{a'\}$ iff $b \in \mathscr{L}(\mathscr{H})$, $b \cap a = \mathbf{0}$, $b \cup a = \mathbf{1}$, and an $\alpha \in \mathscr{F}(\mathscr{H})$ exists such that $\alpha \in a$ and $\alpha^{\nu} \in b$ [hence, $b \in \{a'\}$ iff $b \in \mathscr{L}(\mathscr{H})$ and $\phi(b) = \mathbb{1} - \phi(a)$].

D6.11. Let $a, b \in \mathscr{L}(\mathscr{H})$, $a < b$, $a \neq b$. Then we say that b covers a iff $a < c < b$ implies either $c = a$ or $c = b$ [hence, b covers a iff a one-dimensional orthogonal projection operator E on $\mathscr{H}$ exists such that $\phi(a) \cap E = \mathbb{0}$ and $\phi(a) \cup E = \phi(b)$]. We say that $a \in \mathscr{L}(\mathscr{H})$ is an *atom* iff a covers $\mathbf{0}$ (hence, iff $a = [E]_{\sim}$, with E a projection operator on $\mathscr{H}$ whose range is one-dimensional).

Furthermore, the following propositions can easily be proved, and immediately imply that the statements corresponding to axioms C, P, and A of Section 4.2 hold in our model.

P6.5. For every $a \in \mathscr{L}(\mathscr{H})$,

$$b = \{\beta \in \mathscr{F}(\mathscr{H}) | E_{\beta}(\{1\}) = \mathbb{1} - E_{\alpha}(\{1\})\} \in \{a'\}$$

P6.6. For every $a, b \in \mathscr{L}(\mathscr{H})$, $\phi(a) < \phi(b)$ iff $\mathbb{1} - \phi(b) < \mathbb{1} - \phi(a)$ [hence, for every $a \in \mathscr{L}(\mathscr{H})$, $\{a'\}$ is a singleton, i.e., $\{a'\} = \{b\}$, with b defined as in P6.5].

P6.7. (i) For every $a \in \mathscr{L}(\mathscr{H})$, $a \neq \mathbf{0}$, a projection operator E whose range is one-dimensional exists such that $E < \phi(a)$.
(ii) Let $a \in \mathscr{L}(\mathscr{H})$, and let E be a projection operator on $\mathscr{H}$, whose range is one-dimensional, such that $\phi(a) \cap E = \mathbf{0}$; then, $a \cup [E]_{\sim}$ covers a.

Finally, we recall that in Section 4.2, axioms C, P, and A are followed by P4.4, P4.5, which formally state that the lattice $(\mathscr{L}(\mathscr{H}), <)$ is orthocomplemented and weakly modular. These properties of $(\mathscr{L}(\mathscr{H}), <)$ trivially hold in our present model because of the aforesaid (order) isomorphism between $(\mathscr{L}(\mathscr{H}), <)$ and $(\mathscr{E}(\mathscr{H}), <)$.

6.3. We would like to close the present section with a brief discussion of some peculiar features of the JP structures (see Section 3.4) which have not been understood, or which have been believed "pathological," by some of Piron's opponents, and which are clearly illustrated by our model above.

In particular, let us recall the following statements by Hadjisavvas *et al.* (1980), which hold (suitably formalized) in the fqps.

(i) For any $a \in \mathscr{L}\backslash\{\mathbb{1}\}$ there exists some question $\alpha \in a$ such that $[\alpha^{\nu}]_{\sim} \notin a'$.

(ii) Let α_i, $\beta_i \in \mathscr{Q}$, $a, b \in \mathscr{L}$, and $\alpha_i \in a$, $\beta_i \in b$, with $a \neq b$. Then, the compatible complement of $a \cap b$ contains none of the negations $(\alpha_i \prod \beta_i)^{\nu}$ of the questions $\alpha_i \prod \beta_i$ by help of which the proposition $a \cap b$ can be defined.

In the framework of our mathematical model these statements can easily be proved to hold. Indeed, let us consider (i). Whenever $\alpha \neq E_{\alpha}(\{1\})$, the "compatible complement" $b = \{\beta \in \mathscr{F}(\mathscr{H}) | E_{\beta}(\{1\}) = \mathbb{1} - E_{\alpha}(\{1\})\}$ of $a = [\alpha]_{\sim}$ (see D6.10 and P6.5) does not contain α^{ν}, i.e., $b \neq [\alpha^{\nu}]_{\sim}$ (yet, $[\alpha]_{\sim}$ contains $E_{\alpha}(\{1\})$, which is such that $b = [E_{\alpha}(\{1\})^{\nu}]_{\sim}$).

Let us now consider (ii). It is apparent from rule 6 in Section 6.2 that $\alpha_i \prod \beta_i$ never is a projection. Hence, $a \cap b = [\alpha_i \prod \beta_i]_{\sim}$ by definition, but

$$(a \cap b)' = [E_{\alpha_i \prod \beta_i}(\{1\})^{\nu}]_{\sim} \neq [(\alpha_i \prod \beta_i)^{\nu}]_{\sim}$$

If we recall the physical meaning of our model discussed in Section 6, we see that (i) and (ii) follow from the existence of questions which are not observables in the usual QP meaning (indeed, only the questions which are represented by orthogonal projections are observables); in particular, $\alpha_i \prod \beta_i$ is not such an observable even if α_i and β_i are.

Thus, the pathological character of (i) and (ii) disappears. The deep roots of the belief that (i) and (ii) are undesirable results lie perhaps in the fact that it has not been sufficiently realized that Piron's rather sophisticated machinery (definition of compatible complement and axioms C, P) is built up so that (1) the subclass $a_c \subseteq a$ of the questions $\alpha_c \in a$ such that $[\alpha_c^{\nu}]_{\sim}$ is a complement of a is nonvoid, and (2) for every $\alpha_c, \beta_c \in a_c$, $[\alpha_c^{\nu}]_{\sim} = [\beta_c^{\nu}]_{\sim}$. For, Piron seems well aware (at least in his later papers) that a_c does not generally coincide with a, nor does the equation $[\alpha^{\nu}]_{\sim} = [\beta^{\nu}]_{\sim}$ hold for every α, $\beta \in a$.

Thus, statement (ii) above can be restated by saying that the product question $\alpha \prod \beta$ of two questions α, β does not belong to the subclass $([\alpha \prod \beta]_{\sim})_c$ of the proposition $[\alpha \prod \beta]_{\sim}$, if $\alpha \in ([\alpha]_{\sim})_c$ or $\beta \in ([\beta]]_{\sim})_c$; therefore, the equation $[\alpha \prod \beta]'_{\sim} = [(\alpha \prod \beta)^{\nu}]_{\sim}$ (equivalently $([\alpha]_{\sim} \cap [\beta]_{\sim})' = [(\alpha \prod \beta)^{\nu}]_{\subset}$) does not hold.

It is undoubtedly true, however, that Piron does not give any physical prescription in order to distinguish the elements of the subclass a_c from the other elements of the proposition a. We have already discussed this subject (Cattaneo and Nisticò, 1986; Garola, 1980, 1985; Garola and Solombrino, 1983; Cattaneo *et al.*, 1986), and we will not repeat the details here. We only recall that, under suitable additional assumptions, the subclass a_c can be identified with the subclass of all the elements $\alpha_c \in a$ such that their "certainly no domain" (i.e., the set of the preparations, which make α_c false) is maximum in a, so that the rule for constructing the compatible complement a' of any proposition a is as follows: consider all the questions in a, choose a question $\alpha_c \in a$ such that its "certainly no domain" is maximum in a, take its inverse question α_c^{ν}, and consider the proposition $[\alpha_c^{\nu}]_{\sim}$; this will be the compatible complement of a.

APPENDIX

With reference to Section 4.2, we exhibit here a formal proof of Theorem 4.1; more precisely, of statement (i) in this theorem, since the proof of (ii) can be easily carried out along similar lines.

Then, the following statement must be proved:

$$(\forall a)(\exists b)((b \in \mathcal{L}) \wedge ((\forall \alpha)((\alpha \in a) \to (b < [\alpha]_{\sim})))$$
$$\wedge ((\forall c)(((c \in \mathcal{L}) \wedge ((\forall \alpha)((\alpha \in a) \to (c < [\alpha]_{\sim})))) \to (c < b))))$$

A1. Preliminary Remarks

(a) In every step of the proof some inferential rules are used, which we quote by means of suitable labels as follows (for sake of brevity, we make use both of primitive and derived inferential rules).

LT	logical theorem (more precisely, a valid wff obtained from a logical theorem by substituting every occurrence of a variable with a term which belongs to the same syntactic category of the variable)
EG	existential generalization
MP	modus ponens
Simp	simplification rule
Con	conjunction rule
Trans	transitivity rule
Repl	replacement rule

Since the replacement rule depends on the condition that the replacing expression be equivalent (or logically equivalent) to the replaced expression, the label Repl will be followed, when needed, by a further label indicating the kind of the logical equivalence, according to the following list.

Exp. exportation
Comm. commutativity
Bic. biconditional
QD quantifier displacement

(b) The inferential steps have often been shortened by omitting some steps or by simultaneously applying two or more inferential rules. Thus, we directly transform quantified wffs whenever no misunderstanding arises, and the EG is applied to universally quantified wffs without a previous application of the *universal instantation* rule.

(c) Some inferential steps in the proof depend on a validity notion which is relative to any nonempty universe, not necessarily to any universe whatsoever; for, as far as applied logic is concerned, a validity breakdown which occurs in the only case of an empty universe is irrelevant.

A2. The Proof

(1) $(\forall a)(\forall x)\left(((\forall \alpha)((\alpha \in a) \rightarrow T(x, \alpha))) \leftrightarrow T\left(x, \prod_a\right)\right)$ Axiom 3 for Simp

(2) $(\forall a)(\forall c)\left(\left((c \in \mathscr{L}) \wedge \left((\forall x)\left(T\left(x, \prod_c\right) \rightarrow T\left(x, \prod_a\right)\right)\right)\right) \rightarrow \left((\forall x)\left(T\left(x, \prod_c\right) \rightarrow T\left(x, \prod_a\right)\right)\right)\right)$ LT

(3) $(\forall x)(\forall \alpha)(T(x, \alpha) \leftrightarrow ((\forall \beta)(((\forall y)(T(y, \alpha) \leftrightarrow T(y, \beta))) \rightarrow T(x, \beta))))$ LT

(4) $(\forall a)\left((\forall x)\left(T\left(x, \prod_a\right) \leftrightarrow T\left(x, \prod_a\right)\right)\right)$ LT

(5) $(\forall \alpha)(\forall \beta(((\forall x)(T(x, \alpha) \leftrightarrow T(x, \beta))) \leftrightarrow ((\forall \gamma)(((\forall x)(T(x, \gamma) \leftrightarrow T(x, \alpha))) \leftrightarrow ((\forall x)(T(x, \gamma) \leftrightarrow T(x, \beta))))))$ LT

(6) $(\forall \alpha)(\forall \beta)((\alpha \sim \beta) \leftrightarrow (\beta \in \{\gamma | \alpha \sim \gamma\}))$ for R_1

(7) $(\forall \alpha)(\forall \beta)((\alpha \sim \beta) \leftrightarrow (\beta \in [\alpha]_{\sim}))$ (6), by Repl for D3.3

(8) $(\forall \alpha)(\forall \beta)((\alpha \sim \beta) \leftrightarrow ((\forall x)(T(x, \alpha) \leftrightarrow T(x, \beta))))$ D3.1, D3.2 by Trans

(10) $(\forall a)(\exists \alpha)\left((\forall x)\left(T\left(x, \prod_a\right) \leftrightarrow T(x, \alpha)\right)\right)$ (4) by EG

(11) $(\forall a)(\exists \alpha)\left(\prod_a \sim \alpha\right)$ (8), (10) by Repl and MP

(12) $(\forall a)(\exists \alpha)\left((\forall \beta)\left(\left(\beta \sim \prod_a\right) \leftrightarrow (\beta \sim \alpha)\right)\right)$ (9), (11) by Repl and MP

(13) $(\forall a)(\exists \alpha)(\forall \beta)\left(\left(\beta \in \left[\prod_a\right]_\sim\right) \leftrightarrow (\beta \in [\alpha]_\sim)\right)$

(7), (12) by Repl and Trans

(14) $(\forall a)(\exists \alpha)\left(\left[\prod_a\right]_\sim = [\alpha]_\sim\right)$ (13) by Repl for R_2

(15) $(\forall a)\left(\left(\left[\prod_a\right]_\sim \in \{b \mid (\exists \alpha)(b = [\alpha]_\sim)\}\right)\right.$

$\left.\leftrightarrow \left((\exists \alpha)\left(\left[\prod_a\right]_\sim = [\alpha]_\sim\right)\right)\right)$ R_1, by Repl

(16) $(\forall a)\left(\left[\prod_a\right]_\sim \in \{b \mid (\exists \alpha)(b = [\alpha]_\sim)\}\right)$ (14), (15) by MP

(17) $(\forall a)\left(\left[\prod_a\right]_\sim \in \mathscr{L}\right)$ (16) by Repl for D3.5

(18) $(\forall \alpha)(\forall \beta)((\beta \in [\alpha]_\sim)$

$\leftrightarrow ((\forall x)(T(x, \alpha) \leftrightarrow T(x, \beta))))$ (7), (8) by Trans

(19) $(\forall x)(\forall \alpha)({}^2T(x, [\alpha]_\sim)$

$\leftrightarrow ((\forall \beta)((\beta \in [\alpha]_\sim) \rightarrow T(x, \beta))))$ D3.4 by Repl

(20) $(\forall x)(\forall \alpha)({}^2T(x, [\alpha]_\sim)$

$\leftrightarrow ((\forall \beta)(((\forall y)(T(y, \alpha) \leftrightarrow T(y, \beta))) \rightarrow T(x, \beta))))$

(18), (19) by Repl

(21) $(\forall x)(\forall \alpha)({}^2T(x, [\alpha]_\sim) \leftrightarrow T(x, \alpha))$ (3), (20) by Trans

(22) $(\forall a)(\forall x)$

$\left(((\forall \alpha)((\alpha \in a) \rightarrow {}^2T(x, [\alpha]_\sim))) \leftrightarrow {}^2T\left(x, \left[\prod_a\right]_\sim\right)\right)$ (1), (21) by Repl

(23) $(\forall a)(\forall x)\left({}^2T\left(x, \left[\prod_a\right]_\sim\right)\right.$

$\left.\to ((\forall \alpha)((\alpha \in a) \to {}^2T(x, [\alpha]_\sim)))\right)$ (22) by Repl for Bic

(24) $(\forall a)(\forall x)((\forall \alpha)\left({}^2T\left(x, \left[\prod_a\right]_\sim\right) \to ((\alpha \in a) \to {}^2T(x, [\alpha]_\sim)))\right)$

(23) by Repl for QD

(25) $(\forall a)(\forall x)\left(\left((\forall \alpha)\left((\alpha \in a) \wedge {}^2T\left(x, \left[\prod_a\right]_\sim\right)\right)\right) \to {}^2T(x, [\alpha]_\sim)\right)$

(24) by Repl for Exp and Comm

(26) $(\forall a)(\forall x)\left((\forall \alpha)\left((\alpha \in a) \to \left({}^2T\left(x, \left[\prod_a\right]_\sim\right) \to {}^2T(x, [\alpha]_\sim)\right)\right)\right)$

(25) by Repl for Exp

(27) $(\forall a)\left((\forall \alpha)\left((\alpha \in a) \to \left((\forall x)\left({}^2T\left(x, \left[\prod_a\right]_\sim\right) \to {}^2T(x, [\alpha]_\sim)\right)\right)\right)\right)$

(26) by Repl for QD

(28) $(\forall a)\left((\forall \alpha)\left((\alpha \in a) \to \left(\left[\prod_a\right]_\sim \prec [\alpha]_\sim\right)\right)\right)$ (27) by Repl for D3.6

(29) $(\forall a)(\forall x)\left({}^2T(x, a) \leftrightarrow T\left(x, \prod_a\right)\right)$ (1) by Repl for D3.4

(30) $(\forall a)(\forall c)\left(\left((\forall x)\left(T\left(x, \prod_c\right) \to T\left(x, \prod_a\right)\right)\right) \leftrightarrow \left((\forall x)\left(T\left(x, \prod_c\right)\right.\right.\right.$

$\left.\left.\left.\to \left((\forall \alpha)\left((\alpha \in a) \to T(x, \alpha)\right)\right)\right)\right)\right)$ (1), (29) by Repl

(31) $(\forall a)(\forall c)\left(\left((\forall x)\left(T\left(x, \prod_c\right) \to T\left(x, \prod_a\right)\right)\right)\right.$

$\leftrightarrow \left((\forall \alpha)(\forall x)\left(T\left(x, \prod_c\right)\right.\right.$

$\left.\left.\left.\to ((\alpha \in a) \to T(x, \alpha))\right)\right)\right)$ (30) by Repl for QD

(32) $(\forall a)(\forall c)\left(\left((\forall x)\left(T\left(x, \prod_c\right) \to T\left(x, \prod_a\right)\right)\right)\right)$

$$\leftrightarrow\left((\forall\alpha)(\forall x)\left(\left((\alpha\in a)\wedge T\left(x,\prod_c\right)\right)\rightarrow T(x,\alpha)\right)\right)\right)$$

(31) by Repl for Exp and Comm

(33) $$(\forall a)(\forall c)\left(\left((\forall x)\left(T\left(x,\prod_c\right)\rightarrow T\left(x,\prod_a\right)\right)\right)\leftrightarrow\left((\forall\alpha)(\forall x)\left((\alpha\in a)\rightarrow\left(T\left(x,\prod_c\right)\rightarrow T(x,\alpha)\right)\right)\right)\right)$$

(32) by Repl for Exp

(34) $$(\forall a)(\forall c)\left(\left((\forall x)\left(T\left(x,\prod_c\right)\rightarrow T\left(x,\prod_a\right)\right)\right)\leftrightarrow\left((\forall\alpha)(\forall x)\left((\alpha\in a)\rightarrow({}^2T(x,c)\rightarrow{}^2T(x,[\alpha]_\sim))\right)\right)\right)$$

(21), (29), (33) by Repl

(35) $$(\forall a)(\forall c)\left(\left((\forall x)\left(T\left(x,\prod_c\right)\rightarrow T\left(x,\prod_a\right)\right)\right)\leftrightarrow\left((\forall\alpha)\left((\alpha\in a)\rightarrow((\forall x)({}^2T(x,c)\rightarrow{}^2T(x,[\alpha]_\sim)))\right)\right)\right)$$

(34) by Repl for QD

(36) $$(\forall a)(\forall c)\left(\left((\forall x)\left(T\left(x,\prod_c\right)\rightarrow T\left(x,\prod_a\right)\right)\right)\leftrightarrow\left((\forall x)\left({}^2T(x,c)\rightarrow{}^2T\left(x,\left[\prod_a\right]_\sim\right)\right)\right)\right)$$

(21), (29) by Repl

(37) $$(\forall a)(\forall c)\left(((c\in\mathscr{L})\wedge((\forall\alpha)((\alpha\in a)\rightarrow((\forall x)({}^2T(x,c)\rightarrow{}^2T(x,[\alpha]_\sim)))))) \leftrightarrow\left((\forall x)\left({}^2T(x,c)\rightarrow{}^2T\left(x,\left[\prod_a\right]_\sim\right)\right)\right)\right)$$

(2), (35), (36) by Repl

(38) $$(\forall a)(\forall c)\left(((c\in\mathscr{L})\wedge((\forall\alpha)((\alpha\in a)\rightarrow(c<[\alpha]_\sim))))\leftrightarrow\left(c<\left[\prod_a\right]_\sim\right)\right)$$

(37) by Repl for D3.6

(39) $$(\forall a)\left(\left(\left[\prod_a\right]_\sim\in\mathscr{L}\right)\wedge\left((\forall\alpha)\left((\alpha\in a)\rightarrow\left(\left[\prod_a\right]_\sim<[\alpha]_\sim\right)\right)\right)\wedge\left((\forall c)\left(((c\in\mathscr{L})\wedge((\forall\alpha)((\alpha\in a)\right.\right.\right.$$

$$\rightarrow (c < [\alpha]_\sim)))) \leftrightarrow \left(c < \left[\prod_a\right]_\sim\right))))$$

(17), (28), (38) by Con and Repl for QD

(40) $(\forall a)((\exists b)((b \in \mathscr{L}) \wedge ((\forall \alpha)((\alpha \in a) \rightarrow (b < [\alpha]_\sim)))$
$\wedge ((\forall c)(((c \in \mathscr{L}) \wedge ((\forall \alpha)((\alpha \in a)$
$\rightarrow (c < [\alpha]_\sim)))) \leftrightarrow (c < b)))))$ (39) by EG ■

REFERENCES

Birkhoff, G., and Von Neumann, J. (1936). The logic of quantum mechanics, *Annals of Mathematics*, **37**, 823.

Braithwaite, R. B. (1953). *Scientific Explanation*, Cambridge University Press, Cambridge.

Braithwaite, R. B. (1960). Models in empirical science, in *Proceedings of the Congress of the International Union for the Logic, Methodology and Philosophy of Science*, E. Nagel *et al.*, eds., Stanford.

Carnap, R. (1956). The methodological character of theoretical concepts, in *Minnesota Studies in the Philosophy of Science*, Vol. 1, H. Feigl, M. Scriven, and G. Maxwell, eds., University of Minneapolis Press, Minneapolis, Minnesota.

Carnap, R. (1958). *Introduction to Symbolic Logic and Its Applications*, Dover, New York.

Carnap, R. (1966). *Philosophical Foundations of Physics*, Basic Books, New York.

Cattaneo, G., and Nisticò, G. (1986). A model of the Jauch–Piron approach to the foundation of quantum mechanics based on preclusivity spaces, preprint DMUC.

Cattaneo, G., Garola, C., and Nisticò, G. (1986). Preparation-effect versus question-preparation structures, preprint DMUC.

Cooke, R. M., and Hilgevoord, J. (1981). A new approach to equivalence in Quantum logic, in *Current Issues in Quantum Logic*, E. G. Beltrametti and B. C. van Frassen, eds., Plenum Press, New York.

Duhem, P. (1914). *La théorie physique: son object et sa structure*, Marcel Rivière, Paris.

Foulis, D. J., and Randall, C. H. (1984). A note on misunderstandings of Piron's axioms for quantum mechanics, *Foundations of Physics*, **14**, 65.

Foulis, D. J., Piron, C., and Randall, C. H. (1983). Realism, operationalism, and quantum mechanics, *Foundations of Physics*, **13**, 813.

Garola, C. (1980). Propositions and orthocomplementation in quantum logic, *International Journal of Theoretical Physics*, **19**, 369.

Garola, C. (1985). Embedding of posets into lattices in quantum logic, *International Journal of Theoretical Physics*, **24**, 423.

Garola, C., and Solombrino, L. (1983). Yes–No experiments and ordered structures in quantum physics, *Nuovo Cimento*, **77B**, 87.

Groenewold, H. J. (1961). The model in physics, in *The concept and the role of the model in mathematics and natural and social sciences*, H. Freudental, ed., Dordrecht.

Hadjisavvas, N., and Thieffine, F. (1984). Piron's axioms for quantum mechanics; a reply to Foulis and Randall, *Foundations of Physics*, **14**, 83.

Hadjisavvas, N., Thieffine, F., Mugur Shacter, M. (1980). Study of Piron's system of questions and propositions, *Foundations of Physics*, **10**, 751.

Hadjisavvas, N., Thieffine, F., and Mugur Shacter, M. (1981). Supplement to a critique of Piron's system of questions and propositions, *Foundations of Physics*, **11**, 645.

Hempel, C. C. (1965). *Aspects of Scientific Explanation*, Free Press, New York.
Hughes, R. I. G. (1981). Realism and quantum logic, in *Current Issues in Quantum Logic*, E. G. Beltrametti and B. C. van Frassen, eds., Plenum Press, New York.
Jauch, J. M. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.
Jauch, J. M. (1971). Foundations of quantum mechanics, in *Foundations of Quantum Mechanics*, B. D'Espagnat, ed., Academic Press, New York.
Jauch, J. M., and Piron, C. (1969). On the structure of quantal proposition systems, *Helvetica Physica Acta*, **42**, 842.
Jauch, J. M., and Piron, C. (1970). What is quantum logic? in *Quanta*, P. G. O. Freund, C. J. Goebel, and Y. Nambu, eds., Chicago University Press, Chicago.
Ludwig, G. (1983). *Foundations of Quantum Mechanics*, Springer-Verlag, New York.
Mackey, G. W. (1963). *The Mathematical Foundations of Quantum Mechanics*, Benjamin, New York.
Mielnik, B. (1976). Quantum mechanics, is it necessarily orthocomplemented?, in *Quantum Mechanics, Determinism, Causality and Particles*, M. Flato, Z. Maric, A. Milojevic, D. Sternheimer, and J. P. Vigier, eds., Reidel, Dordrecht.
Nagel, E. (1961). *The Structure of Science*, Harcourt, Brace and World, New York.
Piron, C. (1964). Axiomatique quantique, *Helvetica Physica Acta*, **37**, 439.
Piron, C. (1972). Survey of general quantum physics, *Foundations of Physics*, **2**, 287 (1972).
Piron, C. (1976a). *Foundations of Quantum Physics*, Benjamin, Reading, Massachusetts.
Piron, C. (1976b). On the foundations of quantum physics, in *Quantum Mechanics, Determinism, Causality and Particles*, M. Flato *et al.*, eds., Reidel, Dordrecht.
Piron, C. (1977). A first lecture in quantum mechanics, in *Quantum Mechanics, a Half Century After*, J. Leite Lopes and M. Paty, eds., Reidel, Dordrecht.
Piron, C. (1978). La description d'un systeme physique et le presupposè de la theorie classique, *Annales de la Fondation L. de Broglie*, **3**, 131.
Piron, C. (1981). Ideal measurement and probability in quantum mechanics, *Erkenntnis*, **16**, 397.
Popper, K. R. (1968). Birkhoff and Von Neumann's interpretation of quantum mechanics, *Nature*, **219**, 682.
Thieffine, F. (1983). Compatible complement in Piron's system and ordinary modal logic, *Nuovo Cimento Lettere* **36**, 377.
Whitehead, A. N., and Russell, B. (1925). *Principia Mathematica*, Cambridge University Press, London.